BDA015 Stavební mechanika 1
9. přednáška

- Průřezové charakteristiky rovinných obrazců
- Steinerova věta
- Poloměr a elipsa setrvačnosti

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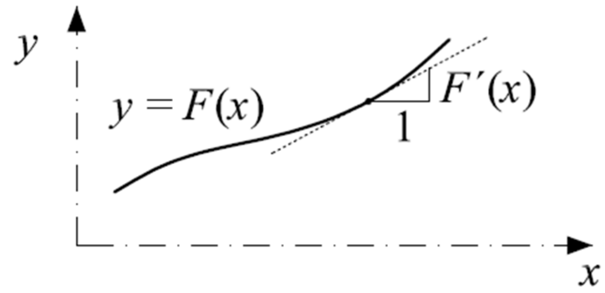
INFINITEZIMÁLNÍ POČET

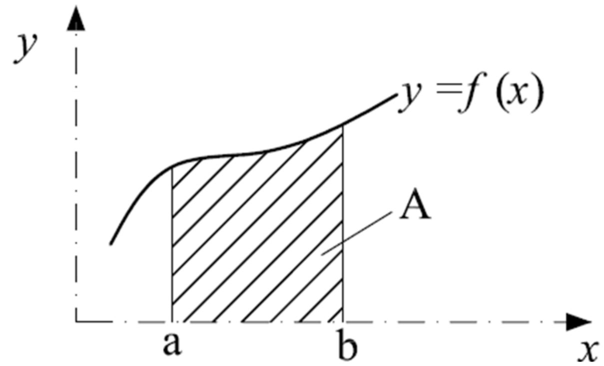
Diferenciální počet → derivace

- zjednodušeně: derivace … sklon (rychlost)
- směrnice tečny v bodě



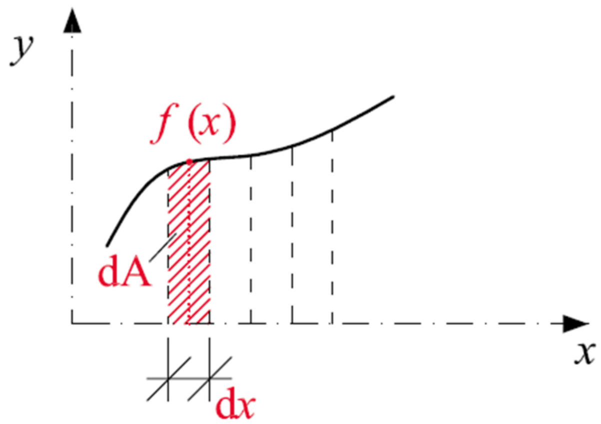
Integrální počet → integrály

- zjednodušeně: integrál … plocha (součet)
- $A = \int_a^b f(x)\,\mathrm{d}x$
- $A \doteq \sum \mathrm{d}A = \sum f(x) \cdot \mathrm{d}x$



Primitivní funkce – souvislost

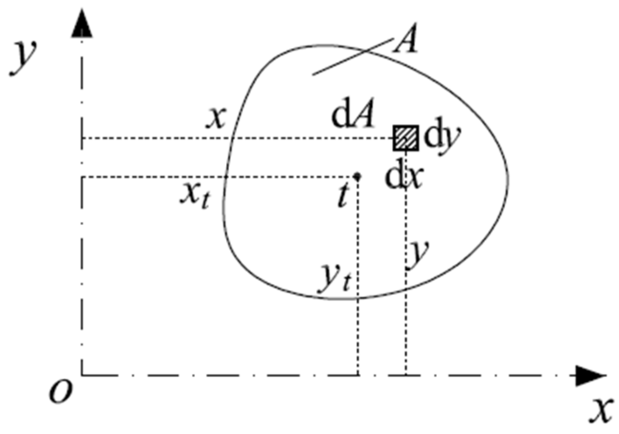
- $f(x) = \frac{\partial F(x)}{\partial x}$
- $F(x) = \int f(x)\mathrm{d}x + c$



TĚŽIŠTĚ ROVINNÝCH OBRAZCŮ

Plocha průřezu *A* [m²]

- $A = \int_A \mathrm{d}A = \iint_A \mathrm{d}x\mathrm{d}y \doteq \sum \mathrm{d}A$

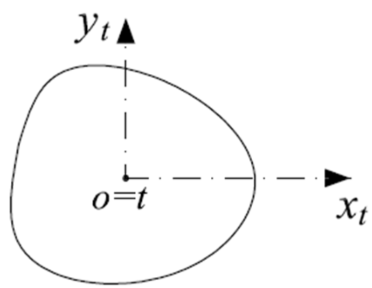


Statický moment plochy k ose *S* [m³]

- $\mathrm{d}S_x = y \cdot \mathrm{d}A;\ \mathrm{d}S_y = x \cdot \mathrm{d}A$
- $S_x = \int_A \mathrm{d}S_x = \int_A y\mathrm{d}A = \iint_A y\,\mathrm{d}x\mathrm{d}y \doteq \sum y\mathrm{d}A$
- $S_y = \int_A \mathrm{d}S_y = \int_A x\mathrm{d}A = \iint_A x\,\mathrm{d}x\mathrm{d}y \doteq \sum x\mathrm{d}A$

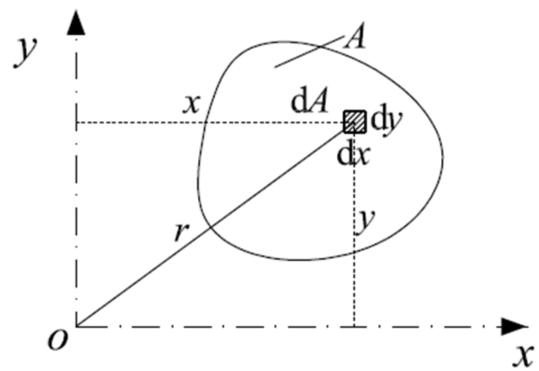
Těžiště [m]

- statický střed soustavy fiktivních rovnoběžných sil
- $x_t = \frac{S_y}{A};\ y_t = \frac{S_x}{A}$
- $S_{x_t} = 0;\ S_{y_t} = 0$



Moment setrvačnosti k ose I [m⁴]

- $I_x = \int_A y^2 \mathrm{d}A = \iint_A y^2 \,\mathrm{d}x\mathrm{d}y (\doteq \sum y^2 \mathrm{d}A)$
- $I_y = \int_A x^2 \mathrm{d}A = \iint_A x^2 \,\mathrm{d}x\mathrm{d}y (\doteq \sum x^2 \mathrm{d}A)$



Polární moment setrvačnosti k bodu I [m⁴]

- $I_o = \int_A r^2 \mathrm{d}A = \int_A (x^2 + y^2) \mathrm{d}A = \int_A y^2 \mathrm{d}A + \int_A x^2 \mathrm{d}A = I_x + I_y$

Deviační moment ke dvěma pravoúhlým osám D [m⁴]

- $D_{xy} = \int_A xy \mathrm{d}A = \iint_A xy \,\mathrm{d}x\mathrm{d}y (\doteq \sum xy \mathrm{d}A)$

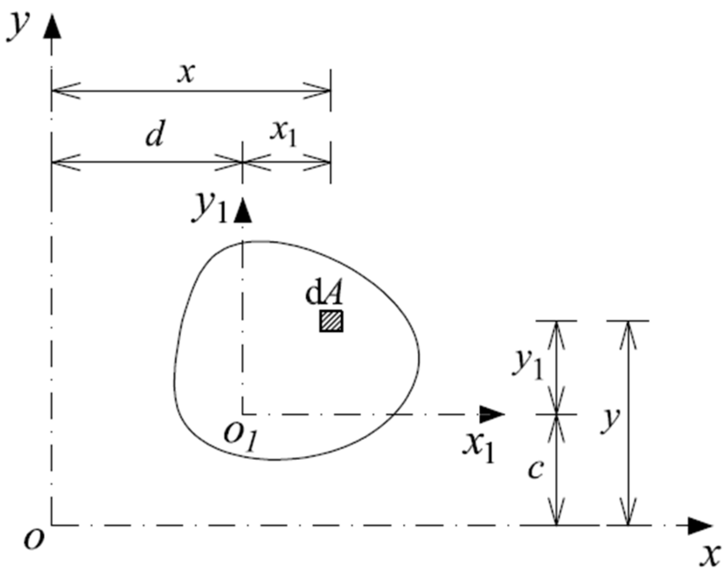
Posunutí souřadnicového systému

- $y = y_1 + \mathrm{c}; x = x_1 + d$
- $I_x = \int_A y^2 \mathrm{d}A = \int_A (y_1 + c)^2 \mathrm{d}A = \int_A (y_1^2 + 2y_1 c + c^2)\mathrm{d}A =$

  $= \int_A y_1^2 \mathrm{d}A + 2\mathrm{c} \int_A y_1 \mathrm{d}A + c^2 \int_A \mathrm{d}A \rightarrow \boldsymbol{I_x = I_{x_1} + 2cS_{x_1} + c^2 A}$
- $I_y = \int_A x^2 \mathrm{d}A = \int_A (x_1 + d)^2 \mathrm{d}A \rightarrow \boldsymbol{I_y = I_{y_1} + 2dS_{y_1} + d^2 A}$
- $y_1 = y_t \rightarrow S_{x_1} = S_{x_t} = 0$

  $\rightarrow \boldsymbol{I_x = I_{x_t} + c^2 A}$ (Steinerova věta)
- $x_1 = x_t \rightarrow S_{y_1} = S_{y_t} = 0$

  $\rightarrow \boldsymbol{I_y = I_{y_t} + d^2 A}$
- $\boldsymbol{D_{xy} = D_{x_t y_t} + cdA}$



OBDÉLNÍK

Plocha průřezu

- $A = \int_A \mathrm{d}A = \iint_A \mathrm{d}x\mathrm{d}y = \int_0^h \int_0^b \mathrm{d}x\mathrm{d}y \rightarrow \boldsymbol{A = b \cdot h}$

Statický moment plochy k osám

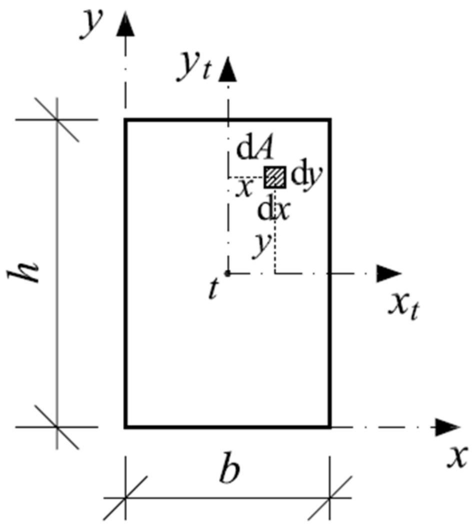
- $S_x = \int_A y\mathrm{d}A = \int_0^h y \int_0^b \mathrm{d}x\mathrm{d}y \rightarrow \boldsymbol{S_x = \frac{bh^2}{2}}$
- $S_y = \int_A x\mathrm{d}A = \int_0^h \int_0^b x\mathrm{d}x\mathrm{d}y \rightarrow \boldsymbol{S_y = \frac{b^2h}{2}}$

Těžiště

- $x_t = \frac{S_y}{A} = \frac{b}{2};\ y_t = \frac{S_x}{A} = \frac{h}{2}$

Statický moment plochy k těžištním osám

- $S_{x_t} = \int_A y\mathrm{d}A = \int_{-h/2}^{h/2} y \int_{-b/2}^{b/2} \mathrm{d}x\mathrm{d}y = b\left[\frac{y^2}{2}\right]_{-h/2}^{h/2} \rightarrow \boldsymbol{S_{x_t} = 0}$
- $S_{y_t} = \int_A x\mathrm{d}A = \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} x\mathrm{d}x\mathrm{d}y \rightarrow \boldsymbol{S_{y_t} = 0}$

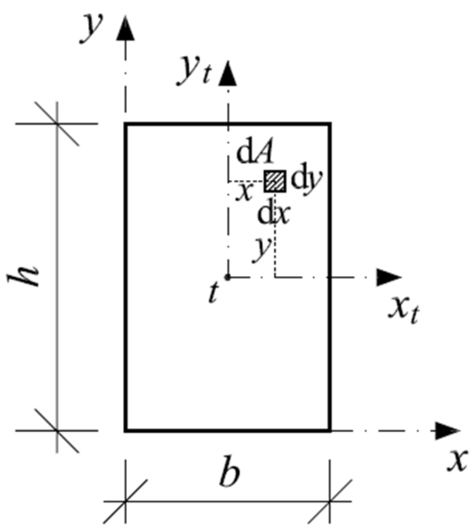


OBDÉLNÍK

Moment setrvačnosti k osám

- $I_x = \int_A y^2 \mathrm{d}A = \iint_A y^2 \,\mathrm{d}x\mathrm{d}y = \int_0^h y^2 \int_0^b \mathrm{d}x\mathrm{d}y = b\left[\frac{y^3}{3}\right]_0^h \rightarrow \boldsymbol{I_x = \frac{bh^3}{3}}$
- $I_y = \int_A x^2 \mathrm{d}A = \iint_A x^2 \,\mathrm{d}x\mathrm{d}y = \int_0^h \int_0^b x^2 \mathrm{d}x\mathrm{d}y \rightarrow \boldsymbol{I_y = \frac{hb^3}{3}}$
- $I_{x_t} = \int_A y^2 \mathrm{d}A = \int_{-h/2}^{h/2} y^2 \int_{-b/2}^{b/2} \mathrm{d}x\mathrm{d}y = b\left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = b\left(\frac{h^3}{24} - \left(-\frac{h^3}{24}\right)\right)$

  $\rightarrow \boldsymbol{I_{x_t} = \frac{1}{12} bh^3}$



Transformace

- $I_{y_t} = \int_A y^2 \mathrm{d}A = I_y + 2dS_y + d^2\mathrm{A} =$

  $= \frac{hb^3}{3} + 2 \cdot \left(-\frac{1}{2}b\right) \cdot \frac{b^2 h}{2} + \left(-\frac{b}{2}\right)^2 \cdot b \cdot h =$

  $= hb^3\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right) \rightarrow \boldsymbol{I_{y_t} = \frac{1}{12} hb^3}$

OBDÉLNÍK

Deviační moment

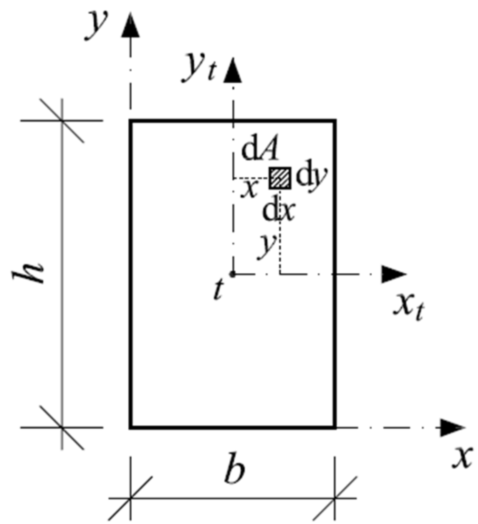
- $D_{xy} = \int_A xy\mathrm{d}A = \iint_A xy\,\mathrm{d}x\mathrm{d}y = \int_0^h y \int_0^b x\mathrm{d}x\mathrm{d}y \rightarrow \boldsymbol{D_{xy} = \frac{b^2h^2}{4}}$

Deviační moment k těžištním osám

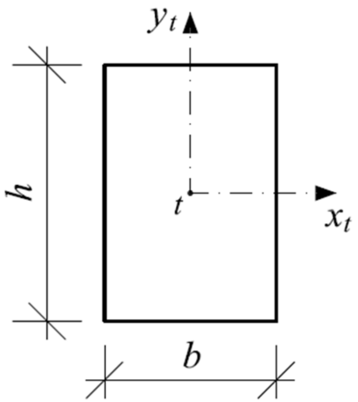
- $D_{x_ty_t} = \int_A xy\mathrm{d}A = \iint_A xy\,\mathrm{d}x\mathrm{d}y = \int_{-h/2}^{h/2} y \int_{-b/2}^{b/2} x\mathrm{d}x\mathrm{d}y \rightarrow$
  $\rightarrow \boldsymbol{D_{x_ty_t} = 0}$

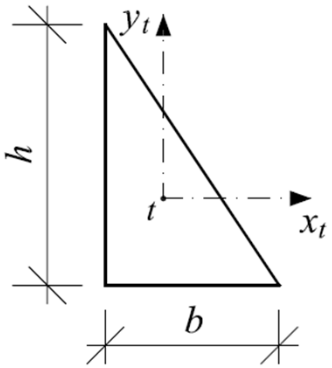
Polární moment setrvačnosti k těžišti

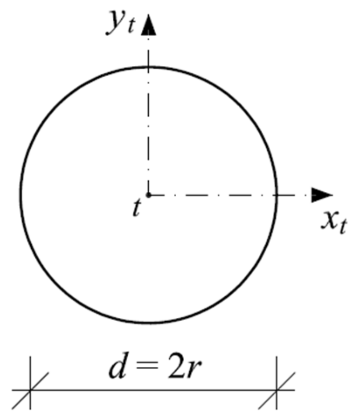
- $I_t = I_{x_t} + I_{y_t} = \frac{1}{12}bh^3 + \frac{1}{12}hb^3 \rightarrow \boldsymbol{I_t = \frac{bh}{12}\left(h^2 + b^2\right)}$



Plocha a centrální kvadratické momenty







$$A = bh$$

$$I_{x_t} = \frac{1}{12}bh^3$$

$$I_{y_t} = \frac{1}{12}hb^3$$

$$D_{x_t y_t} = 0$$

$$A = \frac{1}{2}bh$$

$$I_{x_t} = \frac{1}{36}bh^3$$

$$I_{y_t} = \frac{1}{36}hb^3$$

$$D_{x_t y_t} = -\frac{b^2h^2}{72}$$

$$A = \pi r^2 = \frac{\pi d^2}{4}$$

$$I_{x_t} = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

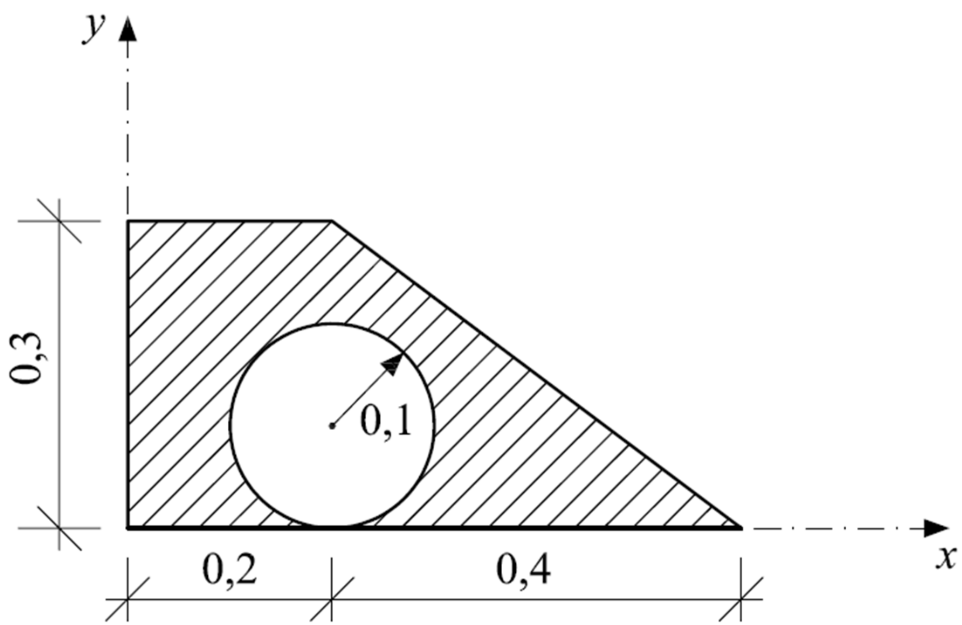
$$I_{y_t} = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

$$D_{x_t y_t} = 0$$

SLOŽENÉ OBRAZCE

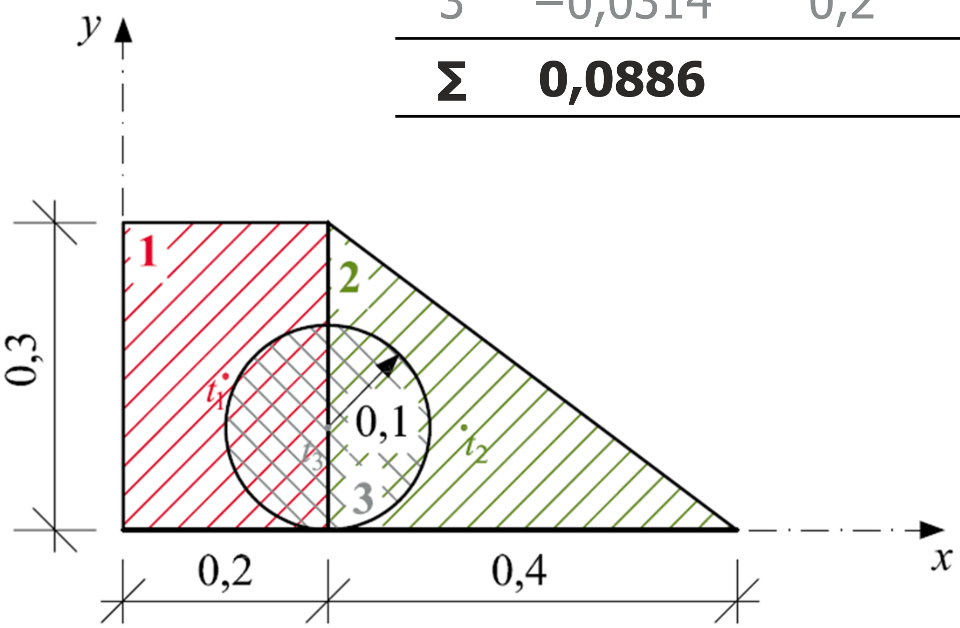
- rozdělení na jednoduché dílčí obrazce
- $A = \sum_{i=1}^{n} A_i$
- $S_x = \sum_{i=1}^{n} S_{x_i} = \sum_{i=1}^{n} A_i \cdot y_i$
- $S_y = \sum_{i=1}^{n} S_{y_i} = \sum_{i=1}^{n} A_i \cdot x_i$
- $x_t = \frac{S_y}{A} = \frac{\sum_{i=1}^{n} A_i \cdot x_i}{A}; \; y_t = \frac{S_x}{A} = \frac{\sum_{i=1}^{n} A_i \cdot y_i}{A}$
- $I_{x_t} = \sum_{i=1}^{n} \left(I_{x_t,i} + A_i \cdot (y_i - y_t)^2 \right)$
- $I_{y_t} = \sum_{i=1}^{n} \left(I_{y_t,i} + A_i \cdot (x_i - x_t)^2 \right)$
- $D_{x_t y_t} = \sum_{i=1}^{n} \left(D_{x_t y_t,i} + A_i \cdot (x_i - x_t) \cdot (y_i - y_t) \right)$

Určete centrální kvadratické momenty daného složeného obrazce



Poloha těžiště

i	A_i[m²]	x_i[m]	y_i[m]	Sx_i[m³]	Sy_i[m³]
1	0,06	0,1	0,15	$9{,}0\cdot10^{-3}$	$6{,}0\cdot10^{-3}$
2	0,06	0,333	0,10	$6{,}0\cdot10^{-3}$	$19{,}98\cdot10^{-3}$
3	−0,0314	0,2	0,1	$-3{,}14\cdot10^{-3}$	$-6{,}28\cdot10^{-3}$
Σ	**0,0886**			$\mathbf{11{,}86\cdot10^{-3}}$	$\mathbf{19{,}70\cdot10^{-3}}$



$$S_x = \sum_{i=1}^{n} S_{x_i} = \sum_{i=1}^{n} A_i \cdot y_i$$

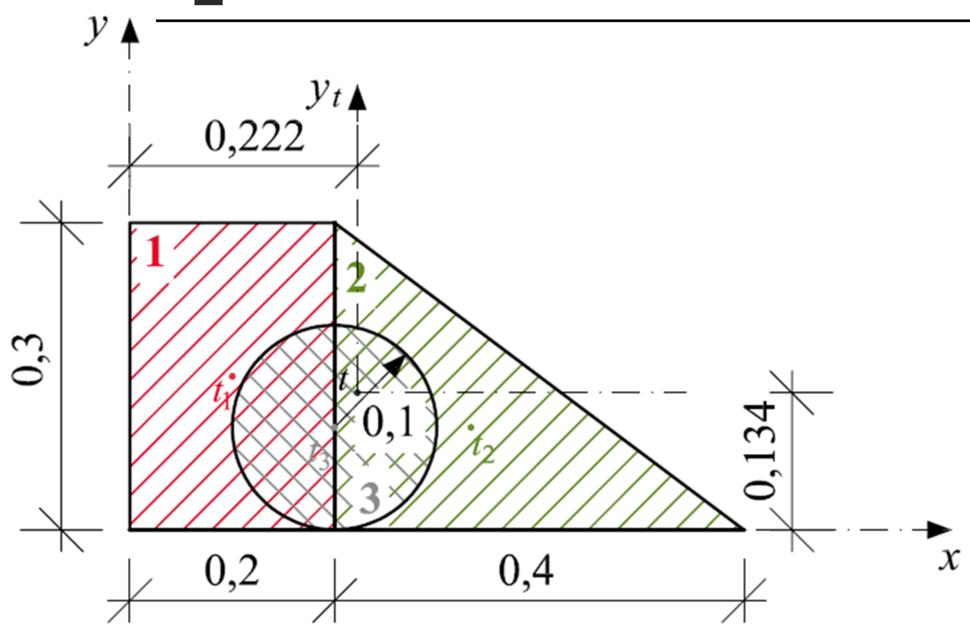
$$S_y = \sum_{i=1}^{n} S_{y_i} = \sum_{i=1}^{n} A_i \cdot x_i$$

$$\boldsymbol{x_t} = \frac{S_y}{A} = \frac{19{,}7\cdot10^{-3}}{0{,}0886} = \mathbf{0{,}222\ m}$$

$$\boldsymbol{y_t} = \frac{S_x}{A} = \frac{11{,}86\cdot10^{-3}}{0{,}0886} = \mathbf{0{,}134\ m}$$

Centrální kvadratické momenty

i	A_i[m²]	y_i[m]	$y_i - y_t$[m]	$Ix_{t,i}$[m⁴]	$A_i(y_i - y_t)^2$[m⁴]
1	0,06	0,15	0,016	$4{,}5 \cdot 10^{-4}$	$1{,}536 \cdot 10^{-5}$
2	0,06	0,10	−0,034	$3{,}0 \cdot 10^{-4}$	$6{,}936 \cdot 10^{-5}$
3	−0,0314	0,1	−0,034	$-0{,}785 \cdot 10^{-4}$	$-3{,}630 \cdot 10^{-5}$
Σ				$\mathbf{6{,}715 \cdot 10^{-4}}$	$\mathbf{0{,}484 \cdot 10^{-4}}$

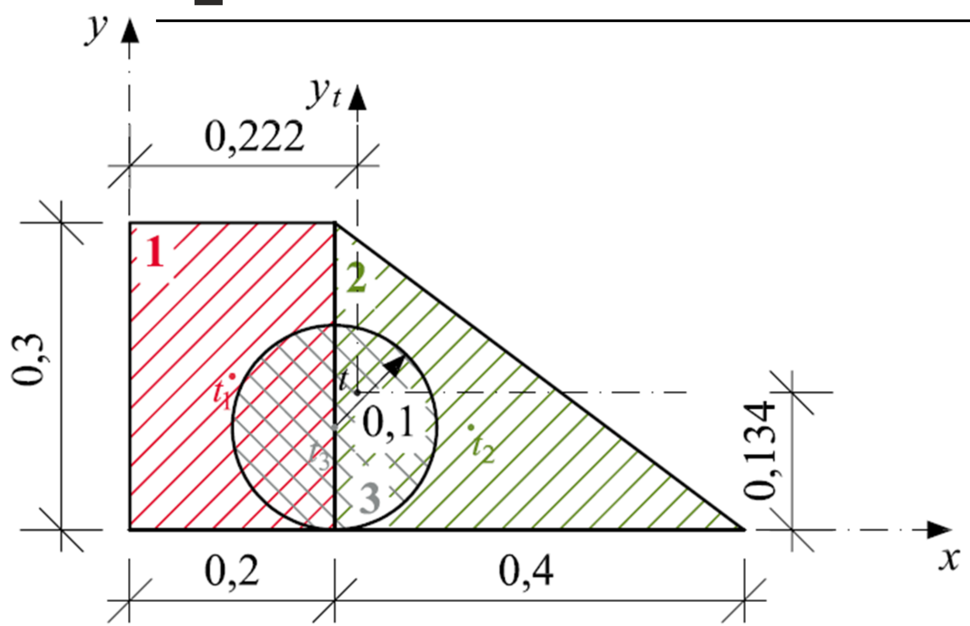


$$I_{x_t} = \sum_{i=1}^{n} \left(I_{x_t,i} + A_i \cdot (y_i - y_t)^2 \right)$$

$$\boldsymbol{I_{x_t} = 7,199 \cdot 10^{-4}\ \mathrm{m}^4}$$

Centrální kvadratické momenty

i	A_i[m²]	x_i[m]	y_i-y_t[m]	$Iy_{t,i}$[m⁴]	$A_i(x_i-x_t)^2$[m⁴]
1	0,06	0,1	0,016	$2{,}0\cdot10^{-4}$	$8{,}930\cdot10^{-4}$
2	0,06	0,333	−0,034	$5{,}333\cdot10^{-4}$	$7{,}393\cdot10^{-4}$
3	−0,0314	0,2	−0,034	$-0{,}785\cdot10^{-4}$	$-0{,}152\cdot10^{-4}$
Σ				**$6{,}548\cdot10^{-4}$**	**$16{,}171\cdot10^{-4}$**

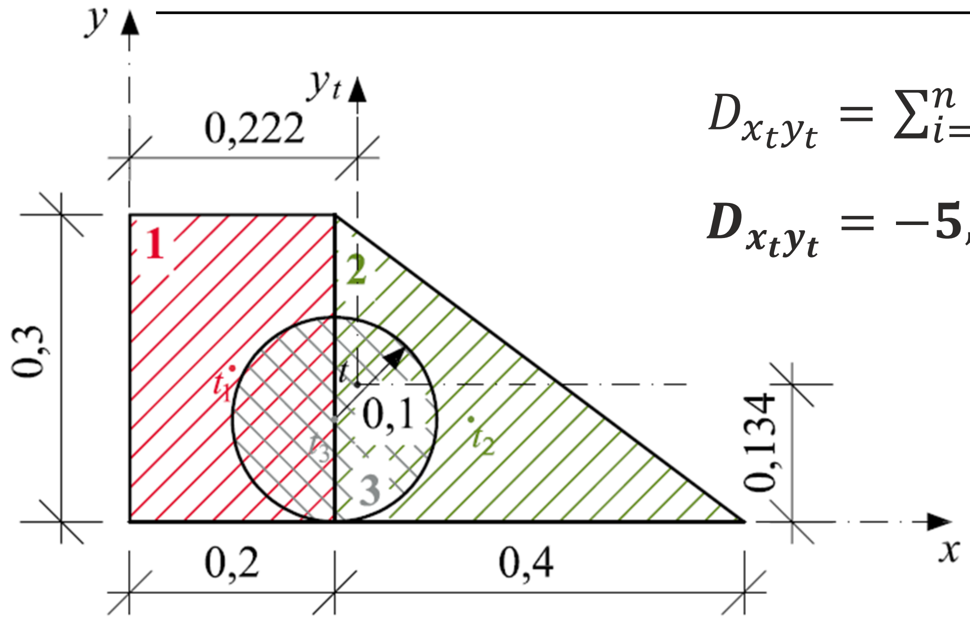


$$I_{y_t} = \sum_{i=1}^{n}\left(I_{y_t,i} + A_i \cdot (x_i - x_t)^2\right)$$

$$\boldsymbol{I_{y_t} = 22{,}719 \cdot 10^{-4}\ \mathrm{m}^4}$$

Centrální kvadratické momenty

i	A_i[m²]	$x_i - x_t$[m]	$y_i - y_t$[m]	$Dx_ty_{t,i}$[m⁴]	$A_i(x_i - x_t)(y_i - y_t)$[m⁴]
1	0,06	−0,122	0,016	0	$-1{,}171 \cdot 10^{-4}$
2	0,06	0,111	−0,034	$-2{,}0 \cdot 10^{-4}$	$-2{,}264 \cdot 10^{-4}$
3	−0,0314	−0,022	−0,034	0	$-0{,}235 \cdot 10^{-4}$
Σ				$\mathbf{-2{,}0 \cdot 10^{-4}}$	$\mathbf{3{,}67 \cdot 10^{-4}}$



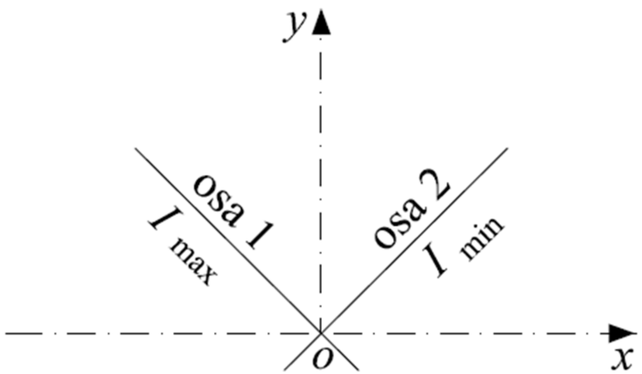
$$D_{x_t y_t} = \sum_{i=1}^{n} \left(D_{x_t y_t,i} + A_i \cdot (x_i - x_t) \cdot (y_i - y_t) \right)$$

$$\boldsymbol{D_{x_t y_t} = -5,67 \cdot 10^{-4}\ \mathrm{m}^4}$$

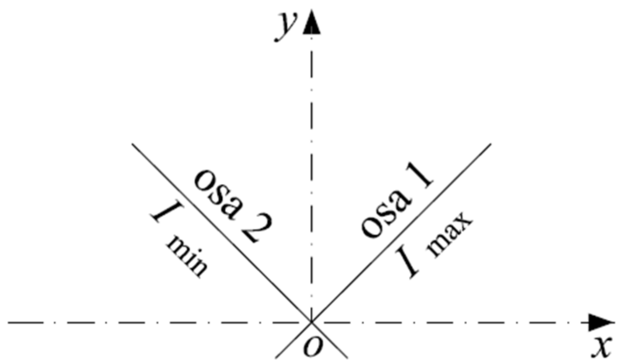
HLAVNÍ OSY SETRVAČNOSTI

- $I_{\max,\min} = I_{1,2} = \frac{I_x + I_y}{2} \pm \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4{D_{xy}}^2}$
- $\operatorname{tg} 2\alpha_0 = \frac{2D_{xy}}{I_y - I_x}$

$D_{xy} > 0$

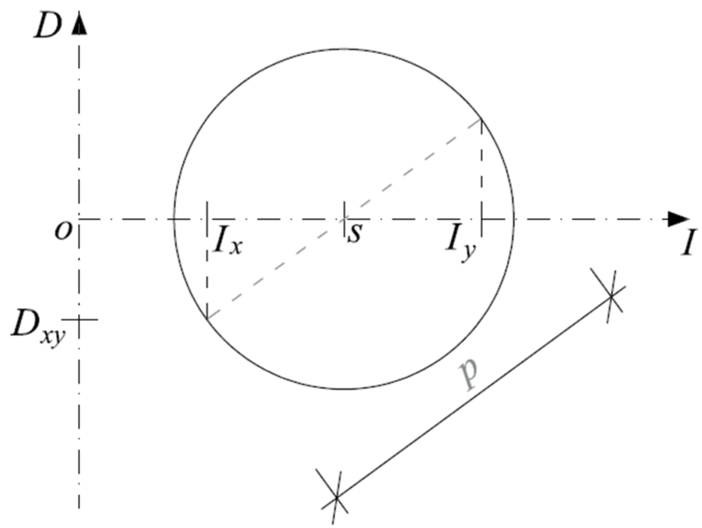


$D_{xy} < 0$



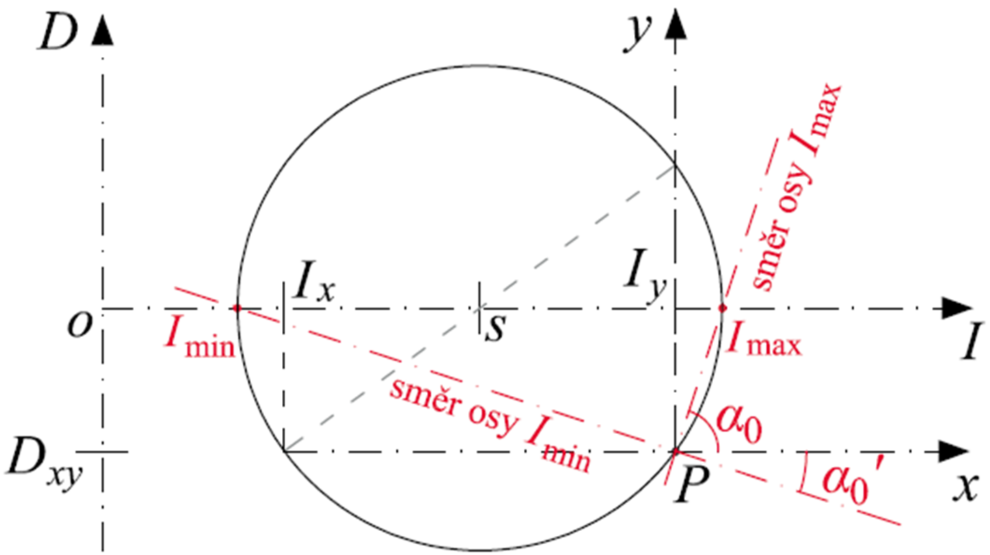
MOHROVA KRUŽNICE

- zvolíme dvě navzájem kolmé osy
  - vodorovná – I_x, I_y
  - svislá – D_{xy}
- ve zvoleném měřítku vyneseme od počátku *o* hodnoty I_x a I_y
- ve stejném měřítku vyneseme kolmo hodnotu D_{xy}
- Mohrova kružnice – kružnice se středem *s* sestrojená nad průměrem *p*
  - $s = \frac{I_x + I_y}{2}$
  - $p = \sqrt{(I_x - I_y)^2 + 4D_{xy}^2}$



MOHROVA KRUŽNICE

- pól Mohrovy kružnice – bod *P* na kružnici, ve kterém se protnou rovnoběžky s osami *x* a *y*
- I_{min}, I_{max} – průsečíky Mohrovy kružnice s vodorovnou osou



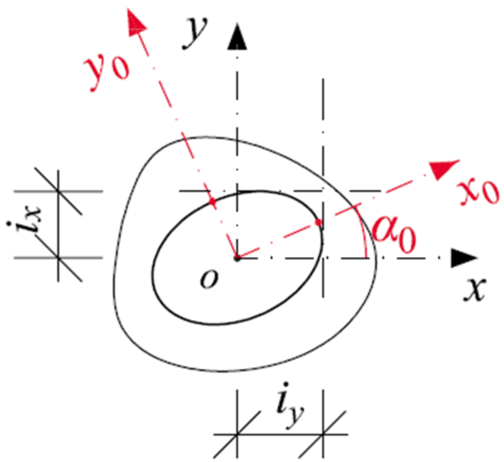
POLOMĚR A ELIPSA SETRVAČNOSTI

Poloměr setrvačnosti [m]

- vzdálenost od těžiště, ve které soustředíme veškerou hmotu a máme stejný moment setrvačnosti jako celého tělesa
- $i_x = \sqrt{\frac{I_x}{A}} \rightarrow I_x = A \cdot i_x^2$; $i_y = \sqrt{\frac{I_y}{A}} \rightarrow I_y = A \cdot i_y^2$

Elipsa setrvačnosti pro daný bod

- každé ose setrvačnosti přísluší moment setrvačnosti a tedy i poloměr setrvačnosti
- přímky rovnoběžné s osami setrvačnosti ve vzdálenostech příslušných poloměrů setrvačnosti obalí křivku – elipsa setrvačnosti



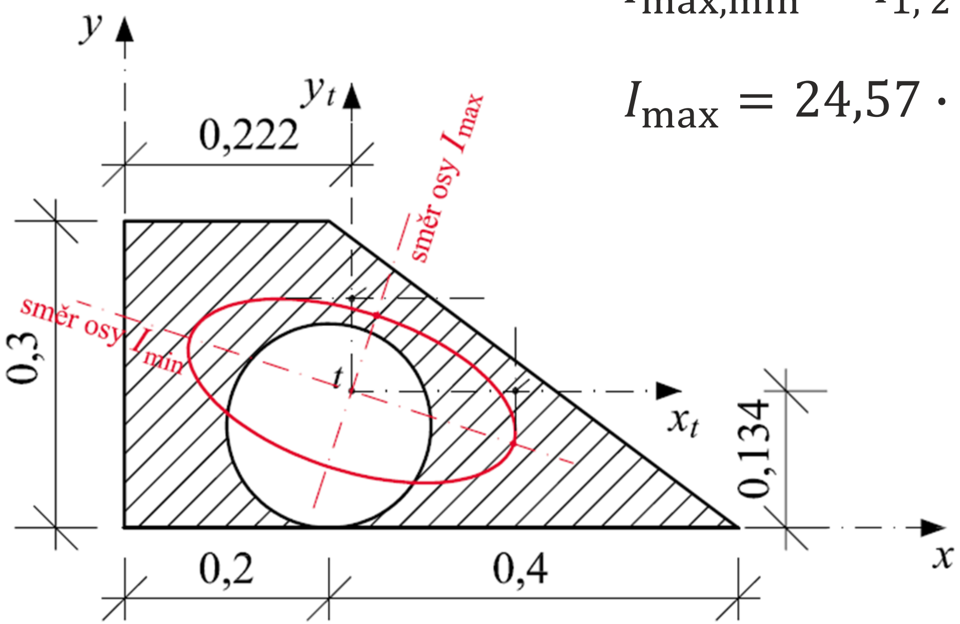
Hlavní centrální kvadratické momenty a elipsa setrvačnosti

$I_{x_t} = 7{,}199 \cdot 10^{-4}\ \mathrm{m}^4;\ I_{y_t} = 22{,}719 \cdot 10^{-4}\ \mathrm{m}^4;\ D_{x_t y_t} = -5{,}67 \cdot 10^{-4}\ \mathrm{m}^4$

$i_{x_t} = \sqrt{\frac{I_{x_t}}{A}} = 90{,}1\ \mathrm{mm};\ i_{y_t} = \sqrt{\frac{I_{y_t}}{A}} = 160{,}1\ \mathrm{mm}$

$$I_{\max,\min} = I_{1,2} = \frac{I_x + I_y}{2} \pm \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4D_{xy}{}^2}$$

$$I_{\max} = 24{,}57 \cdot 10^{-4}\ \mathrm{m}^4;\ I_{\min} = 5{,}35 \cdot 10^{-4}\ \mathrm{m}^4$$

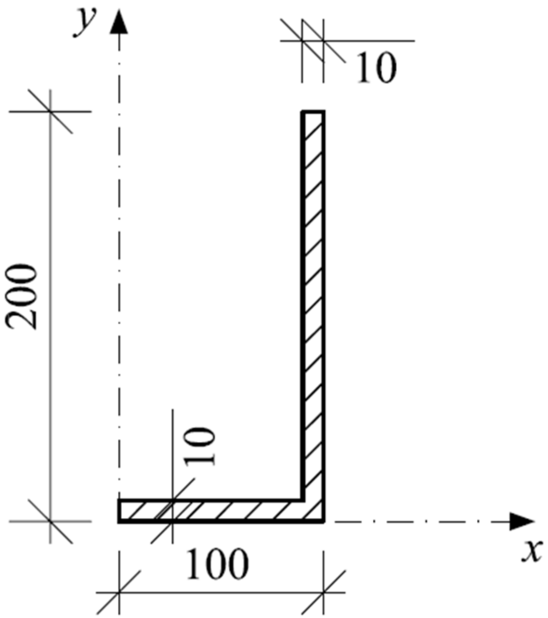


$$i_{\max} = \sqrt{\frac{I_{\max}}{A}} = 166{,}5\ \mathrm{mm}$$

$$i_{\min} = \sqrt{\frac{I_{\min}}{A}} = 77{,}7\ \mathrm{mm}$$

$$\mathrm{tg}\, 2\alpha_0 = \frac{2D_{xy}}{I_y - I_x} \rightarrow \alpha_0 = -18{,}077°$$

Určete hlavní centrální kvadratické momenty a vykreslete elipsu setrvačnosti pro daný složený obrazec

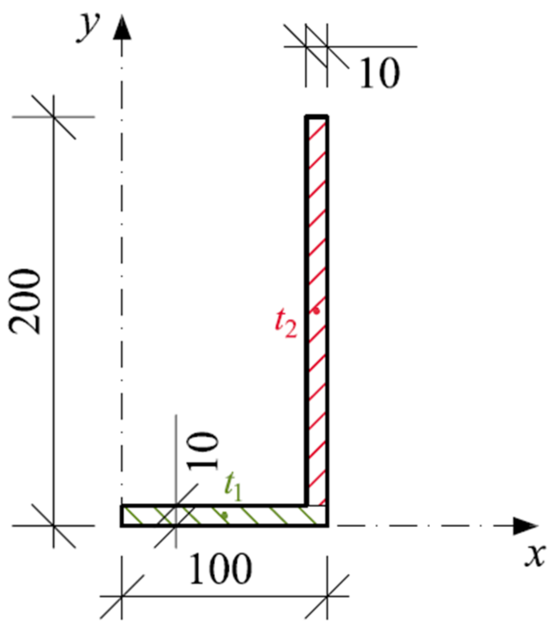


Poloha těžiště

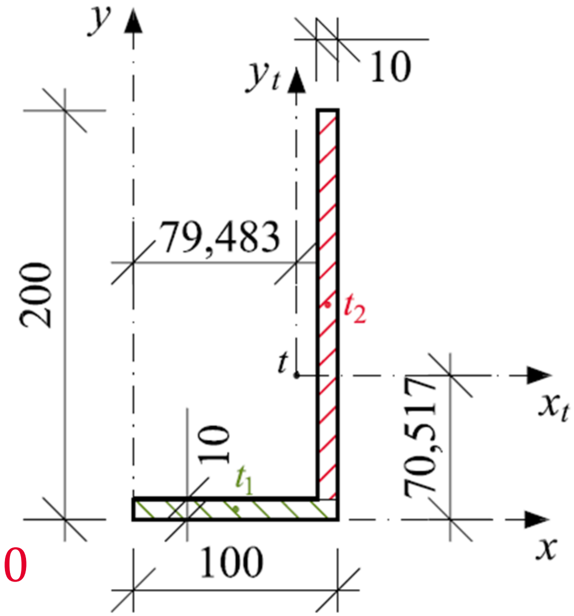
- $\boldsymbol{A} = 100 \cdot 10 + 10 \cdot 190 = \mathbf{2\,900\ mm^2}$
- $\boldsymbol{S_x} = \sum_{i=1}^{n} S_{x_i} = \sum_{i=1}^{n} A_i \cdot y_i =$

  $= 100 \cdot 10 \cdot 5 + 10 \cdot 190 \cdot 105 = \mathbf{204\,500\ mm^3}$
- $\boldsymbol{S_y} = \sum_{i=1}^{n} S_{y_i} = \sum_{i=1}^{n} A_i \cdot x_i$

  $= 100 \cdot 10 \cdot 50 + 10 \cdot 190 \cdot 95 = \mathbf{230\,500\ mm^3}$
- $\boldsymbol{x_t} = \frac{S_y}{A} = \frac{230\,500}{2\,900} = \mathbf{79{,}483\ mm}$
- $\boldsymbol{y_t} = \frac{S_x}{A} = \frac{204\,500}{2\,900} = \mathbf{70{,}517\ mm}$



Centrální kvadratické momenty setrvačnosti



- $\boldsymbol{I_{x_t}} = \sum_{i=1}^{n}\left(I_{x_t,i} + A_i \cdot (y_i - y_t)^2\right)$

  $= \frac{1}{12}100 \cdot 10^3 + 100 \cdot 10 \cdot (5 - 70{,}517)^2 + \frac{1}{12}10 \cdot 190^3$
  $+10 \cdot 190 \cdot (105 - 70{,}517)^2 = \mathbf{12,276 \cdot 10^6\ mm^4}$

- $\boldsymbol{I_{y_t}} = \sum_{i=1}^{n}\left(I_{y_t,i} + A_i \cdot (x_i - x_t)^2\right)$

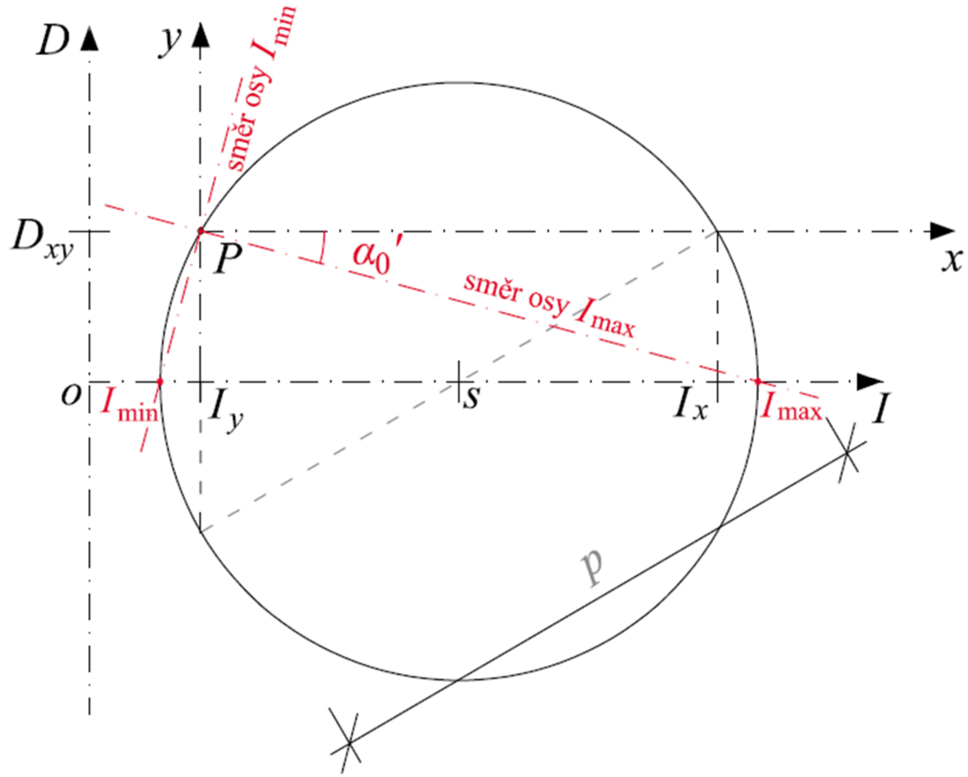
  $= \frac{1}{12}100^3 \cdot 10 + 100 \cdot 10 \cdot (50 - 79{,}483)^2 + \frac{1}{12}10^3 \cdot 190$
  $+10 \cdot 190 \cdot (95 - 79{,}483)^2 = \mathbf{2,176 \cdot 10^6\ mm^4}$

- $\boldsymbol{D_{x_t y_t}} = \sum_{i=1}^{n}\left(D_{x_t y_t,i} + A_i \cdot (x_i - x_t) \cdot (y_i - y_t)\right)$

  $= 0 + 100 \cdot 10 \cdot (50 - 79{,}483) \cdot (5 - 70{,}517) + 0$
  $+10 \cdot 190 \cdot (95 - 79{,}483) \cdot (105 - 70{,}517) = \mathbf{2,948 \cdot 10^6\ mm^4}$

Mohrova kružnice

- $I_{x_t} = 12{,}276 \cdot 10^6\ \mathrm{mm}^4; I_{y_t} = 2{,}176 \cdot 10^6\ \mathrm{mm}^4;\ D_{x_t y_t} = 2{,}948 \cdot 10^6\ \mathrm{mm}^4$
- $s = \frac{I_x + I_y}{2} = 7{,}226 \cdot 10^6\ \mathrm{mm}^4$
- $p = \sqrt{\left(I_x - I_y\right)^2 + 4{D_{xy}}^2}$
  $= 11{,}695 \cdot 10^6\ \mathrm{mm}^4$



Hlavní centrální kvadratické momenty a elipsa setrvačnosti

- $I_{x_t} = 12{,}276 \cdot 10^6\ \mathrm{mm}^4; I_{y_t} = 2{,}176 \cdot 10^6\ \mathrm{mm}^4;\ D_{x_t y_t} = 2{,}948 \cdot 10^6\ \mathrm{mm}^4$
- $i_{x_t} = \sqrt{\frac{I_{x_t}}{A}} = 65{,}062\ \mathrm{mm};\ i_{y_t} = \sqrt{\frac{I_{y_t}}{A}} = 27{,}392\ \mathrm{mm}$
- $I_{\max,\min} = I_{1,\,2} = \frac{I_x + I_y}{2} \pm \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4{D_{xy}}^2}$
- $I_{\max} = 13{,}073 \cdot 10^6\ \mathrm{mm}^4; I_{\min} = 1{,}379 \cdot 10^6\ \mathrm{mm}^4$
- $i_{\max} = \sqrt{\frac{I_{\max}}{A}} = 67{,}141\mathrm{mm}$
- $i_{\min} = \sqrt{\frac{I_{\min}}{A}} = 21{,}806\ \mathrm{mm}$
- $\mathrm{tg}\, 2\alpha_0 = \frac{2D_{xy}}{I_y - I_x} \rightarrow \alpha_0 = -15{,}137°$