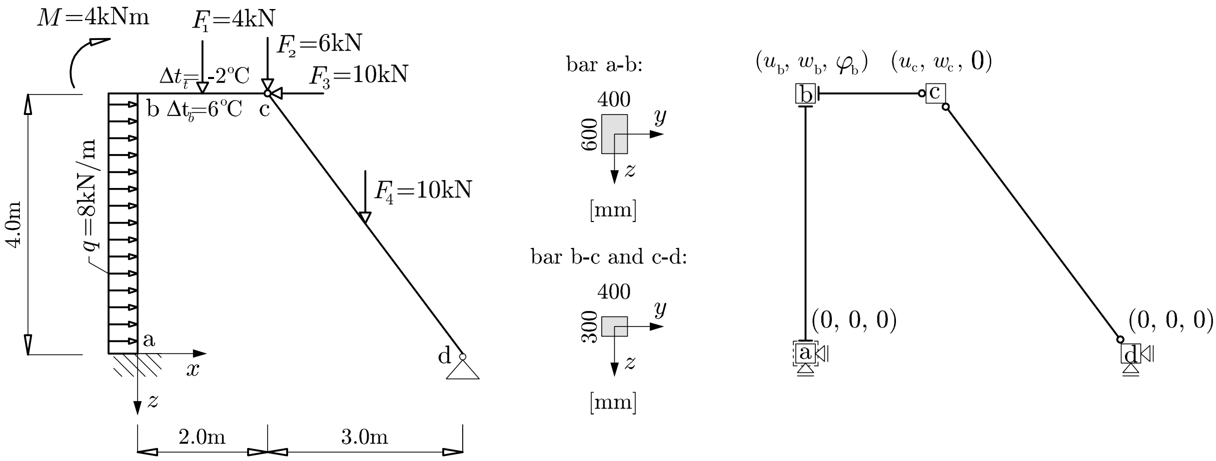
1.2 2D rám

Obecnou deformační metodou vypočítejte a vykreslete vnitřní síly na rovinném rámu na obr. 7. Materiálové parametry jsou: modul pružnosti $E = 24\,\mathrm{GPa}$ a koeficient teplotní roztažnosti $\alpha_t = 10^{-5}\,\mathrm{K}^{-1}$.



Obrázek 7: Vlevo:2D rám a vpravo: výpočtový model.

Průřezová plocha prutů

$$A_{\mathrm{ab}} = b \cdot h = 0.4 \cdot 0.6 = 0.24\,\mathrm{m}^2$$
$$A_{\mathrm{bc}} = A_{\mathrm{cd}} = b \cdot h = 0.4 \cdot 0.3 = 0.12\,\mathrm{m}^2$$

Momenty setrvačnosti

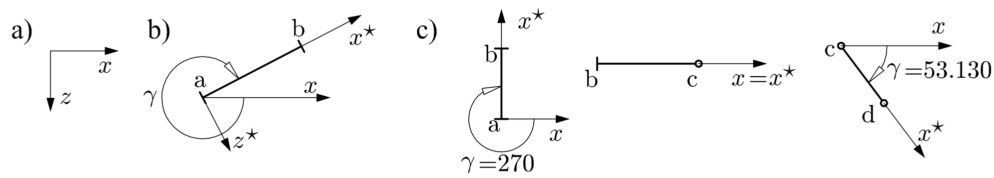
$$I_{\mathrm{y}}^{\mathrm{ab}} = \frac{1}{12} \cdot b \cdot h^3 = \frac{1}{12} \cdot 0.4 \cdot 0.6^3 = 0.0072\,\mathrm{m}^4$$
$$I_{\mathrm{y}}^{\mathrm{bc}} = I_{\mathrm{y}}^{\mathrm{cd}} = \frac{1}{12} \cdot b \cdot h^3 = \frac{1}{12} \cdot 0.4 \cdot 0.3^3 = 0.0009\,\mathrm{m}^4$$

1. stupeň přetvárné neurčitosi

$$n_p = 5$$

2. vektor neznámých deformací a styčníkových sil

$$\{\boldsymbol{r}\} = \begin{Bmatrix} u_{\mathrm{b}} \\ w_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \\ u_{\mathrm{c}} \\ w_{\mathrm{c}} \end{Bmatrix} \qquad \{\boldsymbol{S}\} = \begin{Bmatrix} 0 \\ 0 \\ -4\,000 \\ -10\,000 \\ 6\,000 \end{Bmatrix} [\mathrm{N}, \mathrm{Nm}] \tag{9}$$



Obrázek 8: a) globální souřadný systém xz, b) lokální souřadný systém prutu $x^\star z^\star$ a určení úhlu γ (ve směru hodinových ručiček) c) úhel γ pro pruty a-b, b-c a c-d

3. Globání matice tuhosti

(a) prut a-b — tab. 11.4a, úhel $\gamma = 270° \Rightarrow s = -1, c = 0$

$$[\boldsymbol{k}_{\mathrm{ab}}] =$$

$$\begin{array}{cccccc|c}
u_\mathrm{a} & w_\mathrm{a} & \varphi_\mathrm{a} & u_\mathrm{b} & w_\mathrm{b} & \varphi_\mathrm{b} & \\
\end{array}$$

$$\begin{array}{c} \\ \end{array}\left(\begin{array}{cccccc}
. & . & . & -32.4 & 0 & -64.8 \\
. & . & . & 0 & -1440 & 0 \\
. & . & . & 64.8 & 0 & 86.4 \\
. & . & . & 32.4 & 0 & 64.8 \\
. & . & . & 0 & 1440 & 0 \\
. & . & . & 64.8 & 0 & 172.8
\end{array}\right)\begin{array}{c} u_\mathrm{a} \\ w_\mathrm{a} \\ \varphi_\mathrm{a} \\ u_\mathrm{b} \\ w_\mathrm{b} \\ \varphi_\mathrm{b} \end{array} \left[\cdot 10^6 \tfrac{\mathrm{N}}{\mathrm{m}}\right]$$

(b) prut b-c – lokální souřadný systém je totožný s globálním — můžeme tedy použít tab. 11.3b

$$[\boldsymbol{k}_{\mathrm{bc}}] = [\boldsymbol{k}^\star_{\mathrm{bc}}] =$$

$$\begin{array}{cccccc}
u_\mathrm{b} & w_\mathrm{b} & \varphi_\mathrm{b} & u_\mathrm{c} & w_\mathrm{c} & \varphi_\mathrm{c}
\end{array}$$

$$\left(\begin{array}{cccccc}
1440 & 0 & 0 & -1440 & 0 & 0 \\
0 & 8.1 & -16.2 & 0 & -8.1 & 0 \\
0 & -16.2 & 32.4 & 0 & 16.2 & 0 \\
-1440 & 0 & 0 & 1440 & 0 & 0 \\
0 & -8.1 & 16.2 & 0 & 8.1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)\begin{array}{c} u_\mathrm{b} \\ w_\mathrm{b} \\ \varphi_\mathrm{b} \\ u_\mathrm{c} \\ w_\mathrm{c} \\ \varphi_\mathrm{c} \end{array} \left[\cdot 10^6 \tfrac{\mathrm{N}}{\mathrm{m}}\right]$$

(c) prut c-d — tab. 11.4d, úhel $\gamma = 53.130° \Rightarrow s = \frac{4}{5}, c = \frac{3}{5}$

$$[\boldsymbol{k}_{\mathrm{cd}}] =$$

$$\begin{array}{cccccc}
u_\mathrm{c} & w_\mathrm{c} & \varphi_\mathrm{c} & u_\mathrm{d} & w_\mathrm{d} & \varphi_\mathrm{d}
\end{array}$$

$$\left(\begin{array}{cccccc}
207.36 & 276.48 & . & . & . & . \\
276.48 & 368.64 & . & . & . & . \\
0 & 0 & . & . & . & . \\
-207.36 & -276.48 & . & . & . & . \\
-276.48 & -368.64 & . & . & . & . \\
0 & 0 & . & . & . & .
\end{array}\right)\begin{array}{c} u_\mathrm{c} \\ w_\mathrm{c} \\ \varphi_\mathrm{c} \\ u_\mathrm{d} \\ w_\mathrm{d} \\ \varphi_\mathrm{d} \end{array} \left[\cdot 10^6 \tfrac{\mathrm{N}}{\mathrm{m}}\right]$$

(d) Globální matice tuhosti prutového sytému

$$[\boldsymbol{k}] =$$

$$\begin{array}{ccccc}
u_\mathrm{b} & w_\mathrm{b} & \varphi_\mathrm{b} & u_\mathrm{c} & w_\mathrm{c}
\end{array}$$

$$\left(\begin{array}{ccccc}
32.4 + 1440 & 0 + 0 & 64.8 + 0 & -1440 & 0 \\
0 + 0 & 1440 + 8.1 & 0 + (-16.2) & 0 & -8.1 \\
64.8 + 0 & 0 + (-16.2) & 172.8 + 32.4 & 0 & 16.2 \\
-1440 & 0 & 0 & 1440 + 207.36 & 0 + 276.48 \\
0 & -8.1 & 16.2 & 0 + 276.48 & 8.1 + 368.64
\end{array}\right)\begin{array}{c} u_\mathrm{b} \\ w_\mathrm{b} \\ \varphi_\mathrm{b} \\ u_\mathrm{c} \\ w_\mathrm{c} \end{array} \left[\cdot 10^6 \tfrac{\mathrm{N}}{\mathrm{m}}\right]$$

$$[\boldsymbol{k}] =$$

$$\begin{array}{ccccc}
u_\mathrm{b} & w_\mathrm{b} & \varphi_\mathrm{b} & u_\mathrm{c} & w_\mathrm{c}
\end{array}$$

$$\left(\begin{array}{ccccc}
1472.4 & 0 & 64.8 & -1440 & 0 \\
0 & 1448.1 & -16.2 & 0 & -8.1 \\
64.8 & -16.2 & 205.2 & 0 & 16.2 \\
-1440 & 0 & 0 & 1647.36 & 276.48 \\
0 & -8.1 & 16.2 & 276.48 & 376.74
\end{array}\right)\begin{array}{c} u_\mathrm{b} \\ w_\mathrm{b} \\ \varphi_\mathrm{b} \\ u_\mathrm{c} \\ w_\mathrm{c} \end{array} \left[\cdot 10^6 \tfrac{\mathrm{N}}{\mathrm{m}}\right]$$

4. Lokální primární vektor koncových sil

   (a) prut a-b — tab 11.2a (tab. 14.10 řádek 13)

$$\left\{\bar{\boldsymbol{R}}^{\star}_{ab}\right\} = \begin{Bmatrix} \bar{X}^{\star}_{\mathrm{ab}} \\ \bar{Z}^{\star}_{\mathrm{ab}} \\ \bar{M}^{\star}_{\mathrm{ab}} \\ \bar{X}^{\star}_{\mathrm{ba}} \\ \bar{Z}^{\star}_{\mathrm{ba}} \\ \bar{M}^{\star}_{\mathrm{ba}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -16 \\ 10.\bar{6} \\ 0 \\ -16 \\ -10.\bar{6} \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{10}$$

   – transformace do globálního souřadného systému

$$\left\{\bar{\boldsymbol{R}}_{ab}\right\} = \begin{Bmatrix} \bar{X}_{\mathrm{ab}} \\ \bar{Z}_{\mathrm{ab}} \\ \bar{M}_{\mathrm{ab}} \\ \bar{X}_{\mathrm{ba}} \\ \bar{Z}_{\mathrm{ba}} \\ \bar{M}_{\mathrm{ba}} \end{Bmatrix} = \begin{Bmatrix} -16 \\ 0 \\ 10.\bar{6} \\ -16 \\ 0 \\ -10.\bar{6} \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{11}$$

   (b) prut b-c — síla F_1: tab. 11.2c (tab. 14.11 řádek 2) a teplotní zatížení t: tab. 11.5b

$$\left\{\bar{\boldsymbol{R}}^{\star}_{bc}\right\}_{\mathrm{F}_1} = \left\{\bar{\boldsymbol{R}}_{bc}\right\}_{\mathrm{F}_1} = \begin{Bmatrix} 0 \\ -2.75 \\ 1.5 \\ 0 \\ -1.25 \\ 0 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{12}$$

$$\left\{\bar{\boldsymbol{R}}^{\star}_{bc}\right\}_{\mathrm{t}} = \left\{\bar{\boldsymbol{R}}_{bc}\right\}_{\mathrm{t}} = \begin{Bmatrix} 57.6 \\ -4.32 \\ 8.64 \\ -57.6 \\ 4.32 \\ 0 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{13}$$

$$\left\{\bar{\boldsymbol{R}}^{\star}_{bc}\right\} = \left\{\bar{\boldsymbol{R}}_{bc}\right\} = \begin{Bmatrix} \bar{X}^{\star}_{\mathrm{bc}} \\ \bar{Z}^{\star}_{\mathrm{bc}} \\ \bar{M}^{\star}_{\mathrm{bc}} \\ \bar{X}^{\star}_{\mathrm{cb}} \\ \bar{Z}^{\star}_{\mathrm{cb}} \\ \bar{M}^{\star}_{\mathrm{cb}} \end{Bmatrix} = \begin{Bmatrix} 57.6 \\ -7.07 \\ 10.14 \\ -57.6 \\ 3.07 \\ 0 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{14}$$

   (c) prut c-d — stejné jako prostě podepřený nosník

$$\left\{\bar{\boldsymbol{R}}^{\star}_{cd}\right\} = \begin{Bmatrix} \bar{X}^{\star}_{\mathrm{cd}} \\ \bar{Z}^{\star}_{\mathrm{cd}} \\ \bar{M}^{\star}_{\mathrm{cd}} \\ \bar{X}^{\star}_{\mathrm{dc}} \\ \bar{Z}^{\star}_{\mathrm{dc}} \\ \bar{M}^{\star}_{\mathrm{dc}} \end{Bmatrix} = \begin{Bmatrix} -4 \\ -3 \\ 0 \\ -4 \\ -3 \\ 0 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{15}$$

– transformace do globálních souřadnic

$$\{\bar{\boldsymbol{R}}_{cd}\} = \begin{Bmatrix} \bar{X}_{\mathrm{cd}} \\ \bar{Z}_{\mathrm{cd}} \\ \bar{M}_{\mathrm{cd}} \\ \bar{X}_{\mathrm{dc}} \\ \bar{Z}_{\mathrm{dc}} \\ \bar{M}_{\mathrm{dc}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5 \\ 0 \\ 0 \\ -5 \\ 0 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N,Nm}\right] \tag{16}$$

(d) Primární vektor prutového systému

$$\{\bar{\boldsymbol{R}}\} = \begin{Bmatrix} -16+57.6 \\ 0+(-7.07) \\ -10.\bar{6}+10.14 \\ -57.6+0 \\ 3.07+(-5) \end{Bmatrix} \cdot 10^3 = \begin{Bmatrix} 41.6 \\ -7.07 \\ -0.52\bar{6} \\ -57.6 \\ -1.93 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N,Nm}\right] \tag{17}$$

5. Zatěžovací vektor

$$\{\boldsymbol{F}\} = \{\boldsymbol{S}\} - \{\bar{\boldsymbol{R}}\} = \begin{Bmatrix} 0 \\ 0 \\ -4 \\ -10 \\ 6 \end{Bmatrix} \cdot 10^3 - \begin{Bmatrix} 41.6 \\ -7.07 \\ -0.52\bar{6} \\ -57.6 \\ -1.93 \end{Bmatrix} \cdot 10^3 = \begin{Bmatrix} -41.6 \\ 7.07 \\ -3.47\bar{3} \\ 47.6 \\ 7.93 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N,Nm}\right] \tag{18}$$

6. Soustava rovnic

$$[\boldsymbol{k}] \cdot \{\boldsymbol{r}\} = \{\boldsymbol{F}\} \tag{19}$$

$$\begin{pmatrix} 1472.4 & 0 & 64.8 & -1440 & 0 \\ 0 & 1448.1 & -16.2 & 0 & -8.1 \\ 64.8 & -16.2 & 205.2 & 0 & 16.2 \\ -1440 & 0 & 0 & 1647.36 & 276.48 \\ 0 & -8.1 & 16.2 & 276.48 & 376.74 \end{pmatrix} \cdot 10^6 \cdot \begin{Bmatrix} u_{\mathrm{b}} \\ w_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \\ u_{\mathrm{c}} \\ w_{\mathrm{c}} \end{Bmatrix} = \begin{Bmatrix} -41.6 \\ 7.07 \\ -3.47\bar{3} \\ 47.6 \\ 7.93 \end{Bmatrix} \cdot 10^3$$

$$\begin{Bmatrix} u_{\mathrm{b}} \\ w_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \\ u_{\mathrm{c}} \\ w_{\mathrm{c}} \end{Bmatrix} = \begin{Bmatrix} 39.171 \\ 4.429 \\ -26.783 \\ 67.736 \\ -27.414 \end{Bmatrix} \left[\cdot 10^{-6}\,\mathrm{m,rad}\right]$$

7. Vektor koncových sil prutu

$$\{\boldsymbol{R}_{\mathrm{ab}}\} = \{\bar{\boldsymbol{R}}_{\mathrm{ab}}\} + \left\{\hat{\boldsymbol{R}}_{\mathrm{ab}}\right\} = \{\bar{\boldsymbol{R}}_{\mathrm{ab}}\} + [\boldsymbol{k}_{\mathrm{ab}}]\{\boldsymbol{r}_{\mathrm{ab}}\} \tag{20}$$

$$\left\{\hat{\boldsymbol{R}}_{\mathrm{ab}}\right\} = \begin{pmatrix} . & . & . & -32.4 & 0 & -64.8 \\ . & . & . & 0 & -1440 & 0 \\ . & . & . & 64.8 & 0 & 86.4 \\ . & . & . & 32.4 & 0 & 64.8 \\ . & . & . & 0 & 1440 & 0 \\ . & . & . & 64.8 & 0 & 172.8 \end{pmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 39.171 \\ 4.429 \\ -26.783 \end{Bmatrix} = \begin{Bmatrix} 466.398 \\ -6377.76 \\ 224.230 \\ -466.398 \\ 6377.76 \\ -2089.822 \end{Bmatrix} [\mathrm{N,Nm}]$$

$$\left\{\hat{\boldsymbol{R}}_{\mathrm{bc}}\right\} = \begin{pmatrix} 1440 & 0 & 0 & -1440 & 0 & 0 \\ 0 & 8.1 & -16.2 & 0 & -8.1 & 0 \\ 0 & -16.2 & 32.4 & 0 & 16.2 & 0 \\ -1440 & 0 & 0 & 1440 & 0 & 0 \\ 0 & -8.1 & 16.2 & 0 & 8.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{Bmatrix} 39.171 \\ 4.429 \\ -26.783 \\ 67.736 \\ -27.414 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -41133.6 \\ 691.813 \\ -1383.626 \\ 41133.6 \\ -691.813 \\ 0 \end{Bmatrix} [\mathrm{N}, \mathrm{Nm}$$

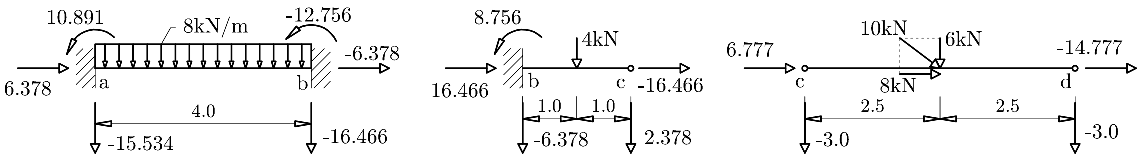
$$\left\{\hat{\boldsymbol{R}}_{\mathrm{cd}}\right\} = \begin{pmatrix} 207.36 & 276.48 & . & . & . & . \\ 276.48 & 368.64 & . & . & . & . \\ 0 & 0 & . & . & . & . \\ -207.36 & -276.48 & . & . & . & . \\ -276.48 & -368.64 & . & . & . & . \\ 0 & 0 & . & . & . & . \end{pmatrix} \cdot \begin{Bmatrix} 67.736 \\ -27.414 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6466.314 \\ 8621.752 \\ 0 \\ -6466.314 \\ -8621.752 \\ 0 \end{Bmatrix} [\mathrm{N}, \mathrm{Nm}]$$

$$\{\boldsymbol{R}_{\mathrm{ab}}\} = \begin{Bmatrix} -15533.602 \\ -6377.76 \\ 10890.89\bar{6} \\ -16466.398 \\ 6377.76 \\ -12765.488\bar{6} \end{Bmatrix} \xrightarrow[\text{transform}]{} \{\boldsymbol{R}^{\star}_{\mathrm{ab}}\} = \begin{Bmatrix} 6377.76 \\ -15533.6 \\ 10890.89\bar{6} \\ -6377.76 \\ -16466.398 \\ -12765.488\bar{6} \end{Bmatrix} [\mathrm{N}, \mathrm{Nm}]$$

$$\{\boldsymbol{R}_{\mathrm{bc}}\} = \{\boldsymbol{R}^{\star}_{\mathrm{bc}}\} = \begin{Bmatrix} 16466.4 \\ -6378.187 \\ 8756.374 \\ -16466.4 \\ 2378.187 \\ 0 \end{Bmatrix} [\mathrm{N}, \mathrm{Nm}]$$

$$\{\boldsymbol{R}_{\mathrm{cd}}\} = \begin{Bmatrix} 6466.314 \\ 3621.752 \\ 0 \\ -6466.314 \\ -13621.752 \\ 0 \end{Bmatrix} \xrightarrow[\text{transform}]{} \{\boldsymbol{R}^{\star}_{\mathrm{cd}}\} = \begin{Bmatrix} 6777.19 \\ -3000 \\ 0 \\ -14777.19 \\ -3000 \\ 0 \end{Bmatrix} [\mathrm{N}, \mathrm{Nm}]$$

8. Kontrola rovnováhy na prutu – obr. 9.



Obrázek 9: Force equilibrium on the bar [kN, kNm]

Bar a – b

$$\sum F_\mathrm{x} = 0 : 6.378 + (-6.378) = 0$$
$$\sum F_\mathrm{z} = 0 : -15.534 + 8 \cdot 4 - 16.466 = 0$$
$$\sum M_\mathrm{a} = 0 : 10.891 - 8 \cdot \frac{4^2}{2} + (-12.756) - (-16.466) \cdot 4 = 0$$

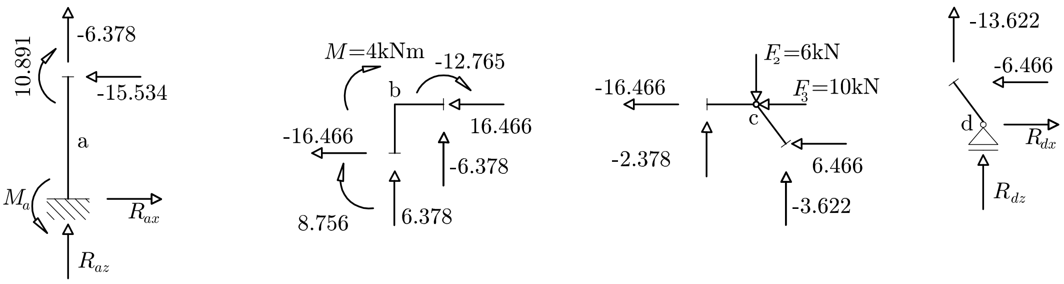
Bar $\mathrm{b-c}$

$$\sum F_\mathrm{x} = 0 : 16.466 + (-16.466) = 0$$
$$\sum F_\mathrm{z} = 0 : -6.378 + 4 + 2.378 = 0$$
$$\sum M_\mathrm{b} = 0 : 8.756 - 4 \cdot 1 - 2.378 \cdot 2 = 0$$

Bar $\mathrm{c-d}$

$$\sum F_\mathrm{x} = 0 : 6.777 + 8 + (-14.777) = 0$$
$$\sum F_\mathrm{z} = 0 : -3 + 6 + (-3) = 0$$
$$\sum M_\mathrm{b} = 0 : 6 \cdot 2.5 + (-3) \cdot 5 = 0$$

9. Kontrola rovnováhy ve styčnících a, b a c a podporových reakcí – obr. 10.



Obrázek 10: Force equilibrium in nodes a, b, c and d [kN, kNm]

Node a

$$\sum F_{xa} = 0 : R_\mathrm{ax} - (-15.534) = 0 \Rightarrow R_\mathrm{ax} = -15.534\,\mathrm{kN}$$
$$\sum F_{za} = 0 : R_\mathrm{az} + (-6.378) = 0 \Rightarrow R_\mathrm{az} = 6.378\,\mathrm{kN}$$
$$\sum M_\mathrm{a} = 0 : M_a - 10.891 = 0 \Rightarrow M_\mathrm{a} = 10.891\,\mathrm{kNm}$$

Node b

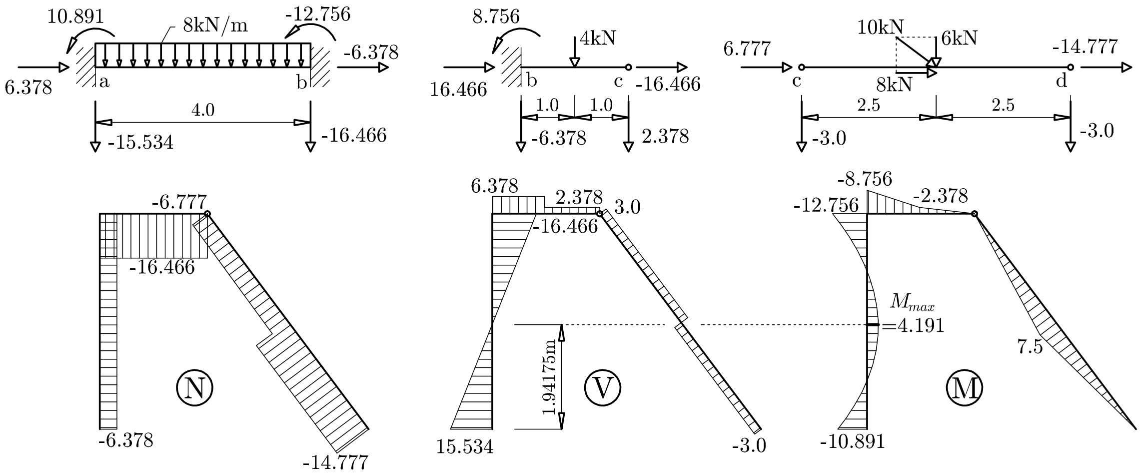
$$\sum F_\mathrm{xb} = 0 : -16.466 + 16.466 = 0$$
$$\sum F_\mathrm{zb} = 0 : 6.378 + (-6.378) = 0$$
$$\sum M_\mathrm{b} = 0 : 8.756 + 4 + (-12.765) = 0$$

Node c

$$\sum F_{xc} = 0 : -16.466 + 10 + 6.466 = 0$$
$$\sum F_\mathrm{zc} = 0 : -2.378 + 6 + (-3.622) = 0$$

Node d

$$\sum F_\mathrm{xd} = 0 : R_\mathrm{dx} - (-6.466) = 0 \Rightarrow R_\mathrm{dx} = -6.466\,\mathrm{kN}$$
$$\sum F_\mathrm{zd} = 0 : R_\mathrm{dz} + (-13.622) = 0 \Rightarrow R_\mathrm{dz} = -13.622\,\mathrm{kN}$$



Obrázek 11: Vnitřní síly rovinného rámu – N, V, M

10. Diagramy vnitřních sil obr. 11.