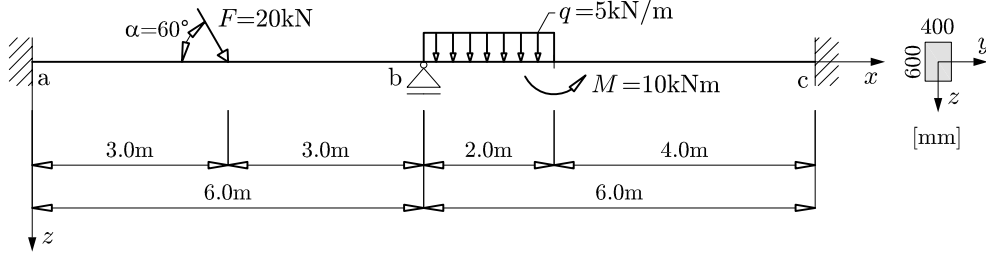


1 Direct Stiffness Method

1.1 Continuous beam

Use Direct Stiffness Method to calculate the internal forces of the continuous beam with rectangular cross-section 0.4×0.6 m and draw them. The beam is at the both end fixed (clamped, built-in, made-in, restrained) and in the middle of the span there is a mobile support. The loading of the beam consists of a single force (point load) 20 kN at angle 60° , uniform load 5 kNm^{-1} in length 2 m and a single moment load 10 kNm (2 meters from the support b). See Fig. 1. The material of this beam has modulus of elasticity $E = 36 \text{ GPa}$.



Obrázek 1: Task scheme

We can write basic equation system:

$$[\mathbf{k}] \cdot \{\mathbf{r}\} = \{\mathbf{F}\} \quad (1)$$

where $[\mathbf{k}]$ = stiffness matrix of the bar system, $\{\mathbf{F}\}$ = force vector, $\{\mathbf{r}\}$ = displacement vector.

Cross-sectional area

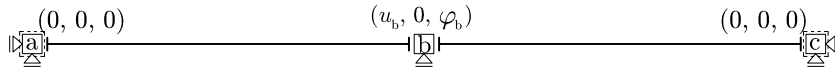
$$A = b \cdot h = 0.4 \cdot 0.6 = 0.24 \text{ m}^2$$

The second moment of area

$$I_y = \frac{1}{12} \cdot b \cdot h^3 = \frac{1}{12} \cdot 0.4 \cdot 0.6^3 = 0.0072 \text{ m}^4$$

1. We label nodes a, b, c and calculate degree of kinematic indeterminacy n_k (degrees of freedom). The kinematic degree of freedom is the number of independent joint displacements (rotations and translations). In a planar task every joint has three degrees of freedom (u_i, w_i, φ_i). One translation in the global X-direction, one translation in the Y-direction and one rotation about Z-axis (counter clockwise). Displacement vector $\{\mathbf{r}\}$ consists of unknown degrees of freedom, in our case $\{\mathbf{r}\} = \{u_b, \varphi_b\}^T \Rightarrow n_k = 2$. We know displacement of some nodes (e.g. $u_a = 0$ node a is supported in X-direction, etc.). See Fig. 2.

$$\{\mathbf{r}\} = \begin{Bmatrix} u_b \\ \varphi_b \end{Bmatrix} \quad (2)$$



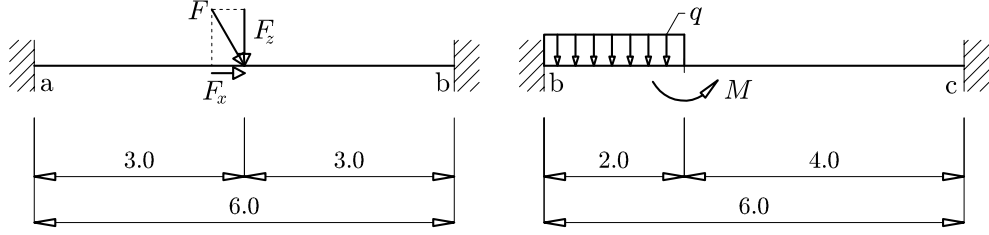
Obrázek 2: Labels of nodes and degrees of freedom

2. We divide our continuous beam into two bars (members) a-b and b-c see Fig. 3.

$$F_x = F \cdot \cos(\alpha) = 20 \cdot \cos(60^\circ) = 10 \text{ kN}$$

$$F_z = F \cdot \sin(\alpha) = 20 \cdot \sin(60^\circ) = 17.32 \text{ kN}$$

To calculate stiffness matrix of members – we use tab. 11.3a, because bars a-b and b-c are fixed supported on both ends. The matrix is symmetric. The stiffness matrices $[\mathbf{k}_{ab}]$ and $[\mathbf{k}_{bc}]$ are identical by reason of the same bar length, cross-sectional area, the second moment of area and modulus of elasticity.



Obrázek 3: Division of the beam into two bars a-b and b-c

$$[\mathbf{k}_{ab}] = [\mathbf{k}_{bc}] = \begin{pmatrix} 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} \left[\cdot 10^9 \frac{\text{N}}{\text{m}} \right]$$

When we solve only displacement vector $\{\mathbf{r}\}$, we need only some matrix elements of $[\mathbf{k}]$ and $\{\bar{\mathbf{R}}\}$. These elements are highlighted below this text and in the next step 3.

$$[\mathbf{k}_{ab}] = \begin{pmatrix} u_a(0) & w_a(0) & \varphi_a(0) & u_b(1) & w_b(0) & \varphi_b(2) \\ 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.0432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} \begin{pmatrix} u_a(0) \\ w_a(0) \\ \varphi_a(0) \\ u_b(1) \\ w_b(0) \\ \varphi_b(2) \end{pmatrix} \left[\cdot 10^9 \frac{\text{N}}{\text{m}} \right]$$

$$[\mathbf{k}_{bc}] = \begin{pmatrix} u_b(1) & w_b(0) & \varphi_b(2) & u_c(0) & w_c(0) & \varphi_c(0) \\ 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.0432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} \begin{pmatrix} u_b(1) \\ w_b(0) \\ \varphi_b(2) \\ u_c(0) \\ w_c(0) \\ \varphi_c(0) \end{pmatrix} \left[\cdot 10^9 \frac{\text{N}}{\text{m}} \right]$$

Using highlighted matrix elements, we will assemble global stiffness matrix of the system

$$[\mathbf{k}] = \left(\begin{array}{cc|cc} 1.44 \cdot 10^9 & 0 & + & 0 \\ +1.44 \cdot 10^9 & +0 & & \\ \hline 0 & 1.728 \cdot 10^8 & & \\ +0 & +1.728 \cdot 10^8 & & \end{array} \right) = \begin{pmatrix} 2.88 \cdot 10^9 & 0 \\ 0 & 3.456 \cdot 10^8 \end{pmatrix} \quad (3)$$

3. Force vector $\{\mathbf{F}\}$ is equal to nodal force vector $\{\mathbf{S}\}$ take away primary force vector $\{\bar{\mathbf{R}}\}$ (the strip above R signify that the vector is primary vector). In our case there is no nodal load this implies $\{\mathbf{S}\} = \emptyset$.

$$\{\mathbf{F}\} = \{\mathbf{S}\} - \{\bar{\mathbf{R}}\} \quad (4)$$

Primary vector $\{\bar{\mathbf{R}}_{ab}\}$ of single force - see tab. 11.2c (14.10 row 2)

$$\{\bar{\mathbf{R}}_{ab}\} = \begin{Bmatrix} \bar{X}_{ab} \\ \bar{Z}_{ab} \\ \bar{M}_{ab} \\ \bar{X}_{ba} \\ \bar{Z}_{ba} \\ \bar{M}_{ba} \end{Bmatrix} = \begin{Bmatrix} -5000 \\ -8660 \\ 12990 \\ -5000 \\ -8660 \\ -12990 \end{Bmatrix} [\text{N}] \quad (5)$$

Primary vector $\{\bar{\mathbf{R}}_{bc}^1\}$ of uniform load in length 2 m - see tab. 14.10(row 11 – formulas are given in the second column)

$$\{\bar{\mathbf{R}}_{bc}^1\} = \begin{Bmatrix} \bar{X}_{bc}^1 \\ \bar{Z}_{bc}^1 \\ \bar{M}_{bc}^1 \\ \bar{X}_{cb}^1 \\ \bar{Z}_{cb}^1 \\ \bar{M}_{cb}^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -9074 \\ 6111 \\ 0 \\ -926 \\ -1667 \end{Bmatrix} \left| \begin{array}{l} 0 \\ -\frac{q \cdot a}{2 \cdot l^3} [2 \cdot l \cdot (l^2 - a^2) + a^3] \\ \frac{q \cdot a^2}{12 \cdot l^2} (6 \cdot b^2 + 3 \cdot a \cdot b + a \cdot l) \\ 0 \\ -\frac{q \cdot a^3}{2 \cdot l^3} (l + b) \\ -\frac{q \cdot a^3}{12 \cdot l^3} (3 \cdot b - l) \end{array} \right. [\text{N}]$$

Primary vector $\{\bar{\mathbf{R}}_{bc}^2\}$ of single moment load - see tab. 11.2d (14.10 row 9)

$$\{\bar{\mathbf{R}}_{bc}^2\} = \begin{Bmatrix} \bar{X}_{bc}^2 \\ \bar{Z}_{bc}^2 \\ \bar{M}_{bc}^2 \\ \bar{X}_{cb}^2 \\ \bar{Z}_{cb}^2 \\ \bar{M}_{cb}^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2222 \\ 0 \\ 0 \\ 2222 \\ 3333 \end{Bmatrix} [\text{N}]$$

Primary vector $\{\bar{\mathbf{R}}_{bc}\}$

$$\{\bar{\mathbf{R}}_{bc}\} = \{\bar{\mathbf{R}}_{bc}^1\} + \{\bar{\mathbf{R}}_{bc}^2\} = \begin{Bmatrix} \bar{X}_{bc} \\ \bar{Z}_{bc} \\ \bar{M}_{bc} \\ \bar{X}_{cb} \\ \bar{Z}_{cb} \\ \bar{M}_{cb} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -11296 \\ 6111 \\ 0 \\ 1296 \\ 1666 \end{Bmatrix} [\text{N}] \quad (6)$$

Force vector $\{\mathbf{F}\}$ see eq. 4. We use only elements in relation to unknown displacements u_b and φ_b .

$$\{\mathbf{F}\} = \{\emptyset\} - \begin{Bmatrix} -5000 + 0 \\ -12990 + 6111 \end{Bmatrix} = \begin{Bmatrix} 5000 \\ 6879 \end{Bmatrix}$$

4. Solution of vector of unknown displacements $\{\mathbf{r}\}$

$$\begin{pmatrix} 2.88 \cdot 10^9 & 0 \\ 0 & 3.456 \cdot 10^8 \end{pmatrix} \cdot \begin{Bmatrix} u_b \\ \varphi_b \end{Bmatrix} = \begin{Bmatrix} 5000 \\ 6879 \end{Bmatrix}$$

$$\begin{Bmatrix} u_b \\ \varphi_b \end{Bmatrix} = \begin{Bmatrix} 1.736 \cdot 10^{-6} \\ 19.905 \cdot 10^{-6} \end{Bmatrix} \begin{bmatrix} \text{m} \\ \text{rad} \end{bmatrix}$$

5. Solution of end reaction vector. $\{\hat{\mathbf{R}}_{bc}\}$ is the secondary force vector. To solve this equation system we need only highlighted columns, which are in relation to displacements differing from 0.

$$\{\mathbf{R}_{ab}\} = \{\bar{\mathbf{R}}_{ab}\} + \{\hat{\mathbf{R}}_{ab}\} = \{\bar{\mathbf{R}}_{ab}\} + [\mathbf{k}_{ab}] \cdot \{\mathbf{r}_{ab}\} \quad (7)$$

$$\{\mathbf{R}_{bc}\} = \{\bar{\mathbf{R}}_{bc}\} + \{\hat{\mathbf{R}}_{bc}\} = \{\bar{\mathbf{R}}_{bc}\} + [\mathbf{k}_{bc}] \cdot \{\mathbf{r}_{bc}\} \quad (8)$$

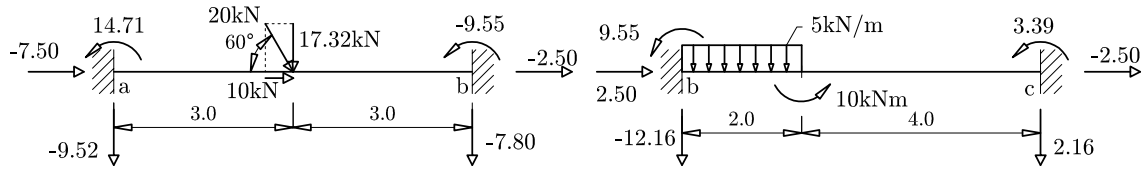
$$\begin{aligned} \{\mathbf{R}_{ab}\} &= \begin{pmatrix} X_{ab} \\ Z_{ab} \\ M_{ab} \\ X_{ba} \\ Z_{ba} \\ M_{ba} \end{pmatrix} = \begin{pmatrix} -5000 \\ -8660 \\ 12990 \\ -5000 \\ -8660 \\ -12990 \end{pmatrix} + \\ &+ \begin{pmatrix} 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.0432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1.736 \\ 0 \\ 19.905 \end{pmatrix} \cdot 10^3 \\ \\ \{\mathbf{R}_{bc}\} &= \begin{pmatrix} X_{bc} \\ Z_{bc} \\ M_{bc} \\ X_{cb} \\ Z_{cb} \\ M_{cb} \end{pmatrix} = \begin{pmatrix} 0 \\ -11296 \\ 6111 \\ 0 \\ 1296 \\ 1666 \end{pmatrix} + \\ &+ \begin{pmatrix} 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.0432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} \begin{pmatrix} 1.736 \\ 0 \\ 19.905 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot 10^3 \end{aligned}$$

There is value 10^3 that we obtain by multiplying stiffness matrix multiplier 10^9 and displacement vector multiplier 10^{-6} .

$$\begin{aligned} \{\mathbf{R}_{ab}\} &= \begin{pmatrix} X_{ab} \\ Z_{ab} \\ M_{ab} \\ X_{ba} \\ Z_{ba} \\ M_{ba} \end{pmatrix} = \begin{pmatrix} -7500.16 \\ -9519.90 \\ 14709.80 \\ -2500.16 \\ -7800.00 \\ -9550.00 \end{pmatrix} [\text{N}, \text{Nm}] \\ \\ \{\mathbf{R}_{bc}\} &= \begin{pmatrix} X_{bc} \\ Z_{bc} \\ M_{bc} \\ X_{cb} \\ Z_{cb} \\ M_{cb} \end{pmatrix} = \begin{pmatrix} 2500 \\ -12156 \\ 9550 \\ -2500 \\ 2156 \\ 3385 \end{pmatrix} [\text{N}, \text{Nm}] \end{aligned}$$

6. Check force equilibrium on the bar – see Fig. 4.

Bar a – b



Obrázek 4: Force equilibrium on the bar [kN, kNm]

$$\sum F_x = 0 : -7.50 + 10 - 2.50 = 0$$

$$\sum F_z = 0 : -9.52 + 17.32 - 7.8 = 0$$

$$\sum M_a = 0 : 14.71 - 17.32 \cdot 3 - (-7.8) \cdot 6 + (-9.55) = 0$$

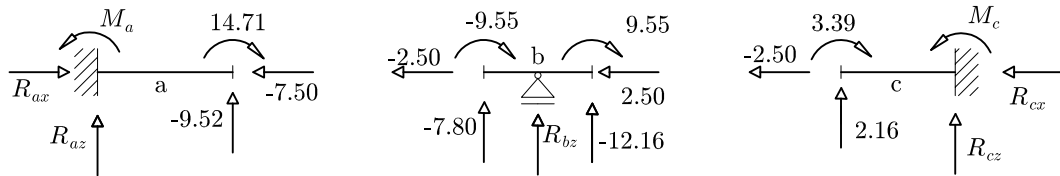
Bar b - c

$$\sum F_x = 0 : 2.50 + (-2.50) = 0$$

$$\sum F_z = 0 : -12.16 + 5 \cdot 2 + 2.16 = 0$$

$$\sum M_b = 0 : 9.55 - \frac{5 \cdot 2^2}{2} + 10 + 3.39 - 2.16 \cdot 6 = 0$$

7. Check force equilibrium in nodes a, b and c – see Fig. 5.



Obrázek 5: Force equilibrium in nodes a, b and c [kN, kNm]

Node a

$$\sum F_{xa} = 0 : R_{ax} - (-7.50) = 0 \Rightarrow R_{ax} = -7.50 \text{ kN}$$

$$\sum F_{za} = 0 : R_{az} + (-9.52) = 0 \Rightarrow R_{az} = 9.52 \text{ kN}$$

$$\sum M_a = 0 : M_c - 14.71 = 0 \Rightarrow M_a = 14.71 \text{ kNm}$$

Node b

$$\sum F_{xb} = 0 : -2.50 + 2.50 = 0$$

$$\sum F_{zb} = 0 : R_{bz} - 7.80 - 12.16 = 0 \Rightarrow R_{bz} = 19.96 \text{ kN}$$

$$\sum M_b = 0 : -9.55 + 9.55 = 0$$

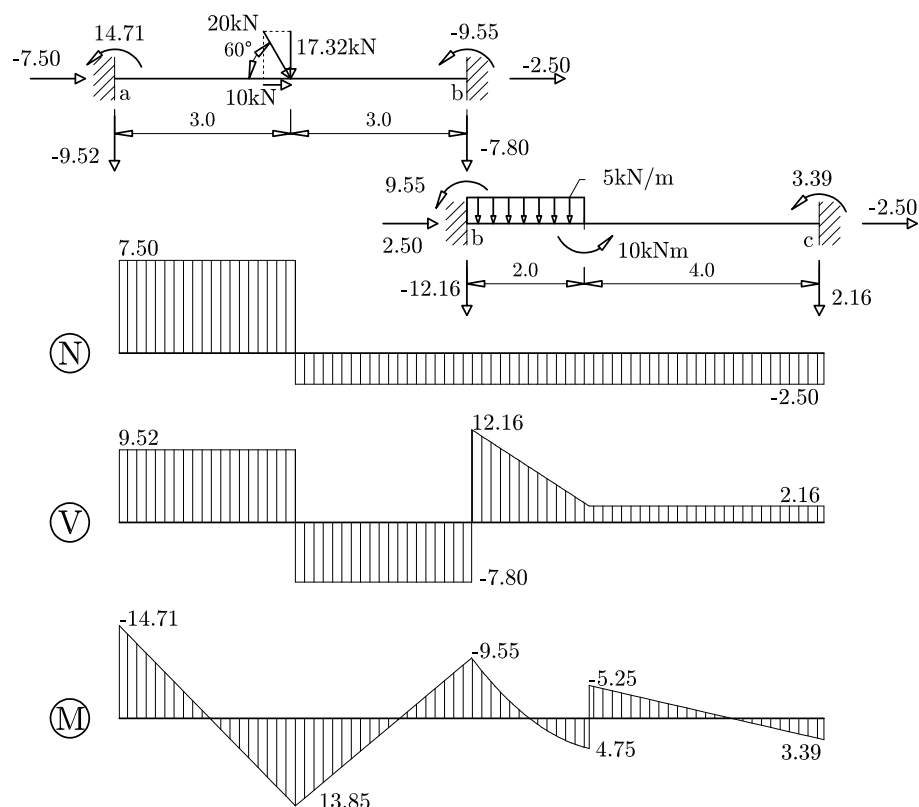
Node c

$$\sum F_{xc} = 0 : R_{cx} - (-2.50) = 0 \Rightarrow R_{cx} = -2.50 \text{ kN}$$

$$\sum F_{zc} = 0 : R_{cz} + 2.16 = 0 \Rightarrow R_{cz} = -2.16 \text{ kN}$$

$$\sum M_c = 0 : M_c - 3.39 = 0 \Rightarrow M_c = 3.39 \text{ kNm}$$

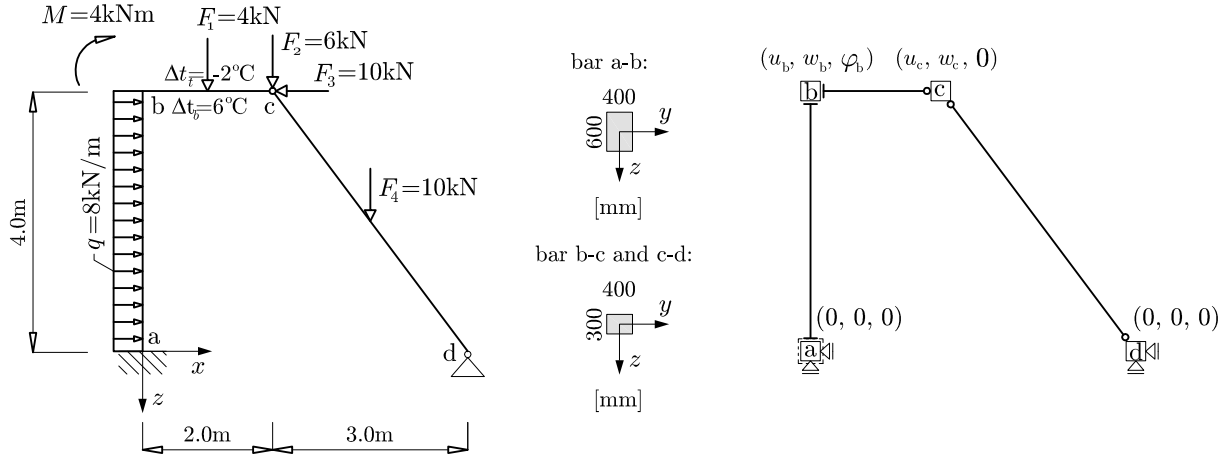
8. Internal forces diagrams (N – normal force, V – shear force, M – bending moment) – see Fig. 6.



Obrázek 6: Internal forces

1.2 2D Frame

Use Direct Stiffness Method to calculate the internal forces of the frame and draw them. See Fig. 7. Material parameters: modulus of elasticity $E = 24 \text{ GPa}$ and coefficient of thermal expansion $\alpha_t = 10^{-5} \text{ K}^{-1}$.



Obrázek 7: Left: 2D frame, right: model scheme.

Cross-sectional area

$$A_{ab} = b \cdot h = 0.4 \cdot 0.6 = 0.24 \text{ m}^2$$

$$A_{bc} = A_{cd} = b \cdot h = 0.4 \cdot 0.3 = 0.12 \text{ m}^2$$

The second moment of area

$$I_y^{ab} = \frac{1}{12} \cdot b \cdot h^3 = \frac{1}{12} \cdot 0.4 \cdot 0.6^3 = 0.0072 \text{ m}^4$$

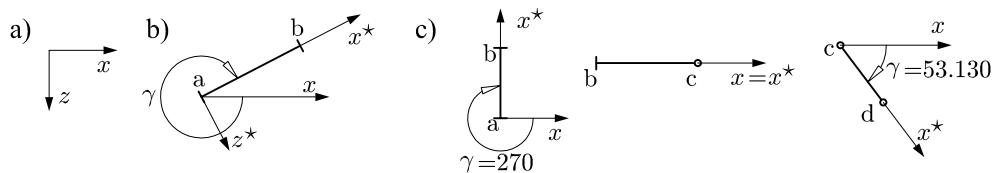
$$I_y^{bc} = I_y^{cd} = \frac{1}{12} \cdot b \cdot h^3 = \frac{1}{12} \cdot 0.4 \cdot 0.3^3 = 0.0009 \text{ m}^4$$

1. degree of kinematic indeterminacy (number of degrees of freedom)

$$n_k = 5$$

2. displacement vector and nodal force vector

$$\{\mathbf{r}\} = \begin{Bmatrix} u_b \\ w_b \\ \varphi_b \\ u_c \\ w_c \end{Bmatrix} \quad \{\mathbf{S}\} = \begin{Bmatrix} 0 \\ 0 \\ -4\,000 \\ -10\,000 \\ 6\,000 \end{Bmatrix} \text{ [N, Nm]} \quad (9)$$



Obrázek 8: a) global coordinate system xz , b) local coordinate system of the general bar x^*z^* and determination of angle γ (clockwise) c) angle γ of our bars a-b, b-c and c-d

3. stiffness matrix in global coordinate system

(a) bar a-b — tab. 11.4a, angle $\gamma = 270^\circ \Rightarrow s = -1, c = 0$

$$[\mathbf{k}_{ab}] = \begin{pmatrix} u_a & w_a & \varphi_a & u_b & w_b & \varphi_b \\ \cdot & \cdot & \cdot & -32.4 & 0 & -64.8 \\ \cdot & \cdot & \cdot & 0 & -1440 & 0 \\ \cdot & \cdot & \cdot & 64.8 & 0 & 86.4 \\ \cdot & \cdot & \cdot & 32.4 & 0 & 64.8 \\ \cdot & \cdot & \cdot & 0 & 1440 & 0 \\ \cdot & \cdot & \cdot & 64.8 & 0 & 172.8 \end{pmatrix} \begin{matrix} u_a \\ w_a \\ \varphi_a \\ u_b \\ w_b \\ \varphi_b \end{matrix} \left[\cdot 10^6 \frac{\text{N}}{\text{m}} \right]$$

(b) bar b-c — local coordinate system of the bar is same as global coordinate of the system — we can use tab. 11.3b

$$[\mathbf{k}_{bc}] = [\mathbf{k}_{bc}^*] = \begin{pmatrix} u_b & w_b & \varphi_b & u_c & w_c & \varphi_c \\ 1440 & 0 & 0 & -1440 & 0 & 0 \\ 0 & 8.1 & -16.2 & 0 & -8.1 & 0 \\ 0 & -16.2 & 32.4 & 0 & 16.2 & 0 \\ -1440 & 0 & 0 & 1440 & 0 & 0 \\ 0 & -8.1 & 16.2 & 0 & 8.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} u_b \\ w_b \\ \varphi_b \\ u_c \\ w_c \\ \varphi_c \end{matrix} \left[\cdot 10^6 \frac{\text{N}}{\text{m}} \right]$$

(c) bar c-d — tab. 11.4d, angle $\gamma = 53.130^\circ \Rightarrow s = \frac{4}{5}, c = \frac{3}{5}$

$$[\mathbf{k}_{cd}] = \begin{pmatrix} u_c & w_c & \varphi_c & u_d & w_d & \varphi_d \\ 207.36 & 276.48 & \cdot & \cdot & \cdot & \cdot \\ 276.48 & 368.64 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ -207.36 & -276.48 & \cdot & \cdot & \cdot & \cdot \\ -276.48 & -368.64 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{matrix} u_c \\ w_c \\ \varphi_c \\ u_d \\ w_d \\ \varphi_d \end{matrix} \left[\cdot 10^6 \frac{\text{N}}{\text{m}} \right]$$

(d) stiffness matrix of the system

$$[\mathbf{k}] = \begin{pmatrix} u_b & w_b & \varphi_b & u_c & w_c \\ 32.4 + 1440 & 0 + 0 & 64.8 + 0 & -1440 & 0 \\ 0 + 0 & 1440 + 8.1 & 0 + (-16.2) & 0 & -8.1 \\ 64.8 + 0 & 0 + (-16.2) & 172.8 + 32.4 & 0 & 16.2 \\ -1440 & 0 & 0 & 1440 + 207.36 & 0 + 276.48 \\ 0 & -8.1 & 16.2 & 0 + 276.48 & 8.1 + 368.64 \end{pmatrix} \begin{matrix} u_b \\ w_b \\ \varphi_b \\ u_c \\ w_c \end{matrix} \left[\cdot 10^6 \frac{\text{N}}{\text{m}} \right]$$

$$[\mathbf{k}] =$$

$$\begin{pmatrix} u_b & w_b & \varphi_b & u_c & w_c \\ 1472.4 & 0 & 64.8 & -1440 & 0 \\ 0 & 1448.1 & -16.2 & 0 & -8.1 \\ 64.8 & -16.2 & 205.2 & 0 & 16.2 \\ -1440 & 0 & 0 & 1647.36 & 276.48 \\ 0 & -8.1 & 16.2 & 276.48 & 376.74 \end{pmatrix} \begin{pmatrix} u_b \\ w_b \\ \varphi_b \\ u_c \\ w_c \end{pmatrix} \quad \left[\cdot 10^6 \frac{\text{N}}{\text{m}} \right]$$

4. primary vectors of end reactions in local coordinates of the bar

(a) bar a-b — tab 11.2a (tab. 14.10 row 13)

$$\{\bar{\mathbf{R}}_{ab}^*\} = \begin{pmatrix} \bar{X}_{ab}^* \\ \bar{Z}_{ab}^* \\ \bar{M}_{ab}^* \\ \bar{X}_{ba}^* \\ \bar{Z}_{ba}^* \\ \bar{M}_{ba}^* \end{pmatrix} = \begin{pmatrix} 0 \\ -16 \\ 10.\bar{6} \\ 0 \\ -16 \\ -10.\bar{6} \end{pmatrix} \left[\cdot 10^3 \text{ N, Nm} \right] \quad (10)$$

– transformation to the global coordinate system

$$\{\bar{\mathbf{R}}_{ab}\} = \begin{pmatrix} \bar{X}_{ab} \\ \bar{Z}_{ab} \\ \bar{M}_{ab} \\ \bar{X}_{ba} \\ \bar{Z}_{ba} \\ \bar{M}_{ba} \end{pmatrix} = \begin{pmatrix} -16 \\ 0 \\ 10.\bar{6} \\ -16 \\ 0 \\ -10.\bar{6} \end{pmatrix} \left[\cdot 10^3 \text{ N, Nm} \right] \quad (11)$$

(b) bar b-c — force F_1 : tab. 11.2c (tab. 14.11 row 2) and thermal load t : tab. 11.5b

$$\{\bar{\mathbf{R}}_{bc}^*\}_{F_1} = \{\bar{\mathbf{R}}_{bc}\}_{F_1} = \begin{pmatrix} 0 \\ -2.75 \\ 1.5 \\ 0 \\ -1.25 \\ 0 \end{pmatrix} \left[\cdot 10^3 \text{ N, Nm} \right] \quad (12)$$

$$\{\bar{\mathbf{R}}_{bc}^*\}_t = \{\bar{\mathbf{R}}_{bc}\}_t = \begin{pmatrix} 57.6 \\ -4.32 \\ 8.64 \\ -57.6 \\ 4.32 \\ 0 \end{pmatrix} \left[\cdot 10^3 \text{ N, Nm} \right] \quad (13)$$

$$\{\bar{\mathbf{R}}_{bc}^*\} = \{\bar{\mathbf{R}}_{bc}\} = \begin{pmatrix} \bar{X}_{bc}^* \\ \bar{Z}_{bc}^* \\ \bar{M}_{bc}^* \\ \bar{X}_{cb}^* \\ \bar{Z}_{cb}^* \\ \bar{M}_{cb}^* \end{pmatrix} = \begin{pmatrix} 57.6 \\ -7.07 \\ 10.14 \\ -57.6 \\ 3.07 \\ 0 \end{pmatrix} \left[\cdot 10^3 \text{ N, Nm} \right] \quad (14)$$

(c) bar c-d — similar to simply supported beam

$$\{\bar{\mathbf{R}}_{cd}^*\} = \begin{Bmatrix} \bar{X}_{cd}^* \\ \bar{Z}_{cd}^* \\ \bar{M}_{cd}^* \\ \bar{X}_{dc}^* \\ \bar{Z}_{dc}^* \\ \bar{M}_{dc}^* \end{Bmatrix} = \begin{Bmatrix} -4 \\ -3 \\ 0 \\ -4 \\ -3 \\ 0 \end{Bmatrix} [\cdot 10^3 \text{ N, Nm}] \quad (15)$$

– transformation to the global coordinate system

$$\{\bar{\mathbf{R}}_{cd}\} = \begin{Bmatrix} \bar{X}_{cd} \\ \bar{Z}_{cd} \\ \bar{M}_{cd} \\ \bar{X}_{dc} \\ \bar{Z}_{dc} \\ \bar{M}_{dc} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5 \\ 0 \\ 0 \\ -5 \\ 0 \end{Bmatrix} [\cdot 10^3 \text{ N, Nm}] \quad (16)$$

(d) primary vector of the system

$$\{\bar{\mathbf{R}}\} = \begin{Bmatrix} -16 + 57.6 \\ 0 + (-7.07) \\ -10.6 + 10.14 \\ -57.6 + 0 \\ 3.07 + (-5) \end{Bmatrix} \cdot 10^3 = \begin{Bmatrix} 41.6 \\ -7.07 \\ -0.526 \\ -57.6 \\ -1.93 \end{Bmatrix} [\cdot 10^3 \text{ N, Nm}] \quad (17)$$

5. force vector

$$\{\mathbf{F}\} = \{\mathbf{S}\} - \{\bar{\mathbf{R}}\} = \begin{Bmatrix} 0 \\ 0 \\ -4 \\ -10 \\ 6 \end{Bmatrix} \cdot 10^3 - \begin{Bmatrix} 41.6 \\ -7.07 \\ -0.526 \\ -57.6 \\ -1.93 \end{Bmatrix} \cdot 10^3 = \begin{Bmatrix} -41.6 \\ 7.07 \\ -3.473 \\ 47.6 \\ 7.93 \end{Bmatrix} [\cdot 10^3 \text{ N, Nm}] \quad (18)$$

6. equation system

$$[\mathbf{k}] \cdot \{\mathbf{r}\} = \{\mathbf{F}\} \quad (19)$$

$$\begin{pmatrix} 1472.4 & 0 & 64.8 & -1440 & 0 \\ 0 & 1448.1 & -16.2 & 0 & -8.1 \\ 64.8 & -16.2 & 205.2 & 0 & 16.2 \\ -1440 & 0 & 0 & 1647.36 & 276.48 \\ 0 & -8.1 & 16.2 & 276.48 & 376.74 \end{pmatrix} \cdot 10^6 \cdot \begin{Bmatrix} u_b \\ w_b \\ \varphi_b \\ u_c \\ w_c \end{Bmatrix} = \begin{Bmatrix} -41.6 \\ 7.07 \\ -3.473 \\ 47.6 \\ 7.93 \end{Bmatrix} \cdot 10^3$$

$$\begin{Bmatrix} u_b \\ w_b \\ \varphi_b \\ u_c \\ w_c \end{Bmatrix} = \begin{Bmatrix} 39.171 \\ 4.429 \\ -26.783 \\ 67.736 \\ -27.414 \end{Bmatrix} [\cdot 10^{-6} \text{ m, rad}]$$

7. End reaction vector of the bar

$$\{\mathbf{R}_{ab}\} = \{\bar{\mathbf{R}}_{ab}\} + \{\hat{\mathbf{R}}_{ab}\} = \{\bar{\mathbf{R}}_{ab}\} + [\mathbf{k}_{ab}] \{\mathbf{r}_{ab}\} \quad (20)$$

$$\{\hat{\mathbf{R}}_{ab}\} = \begin{pmatrix} . & . & . & -32.4 & 0 & -64.8 \\ . & . & . & 0 & -1440 & 0 \\ . & . & . & 64.8 & 0 & 86.4 \\ . & . & . & 32.4 & 0 & 64.8 \\ . & . & . & 0 & 1440 & 0 \\ . & . & . & 64.8 & 0 & 172.8 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 39.171 \\ 4.429 \\ -26.783 \end{pmatrix} = \begin{pmatrix} 466.398 \\ -6377.76 \\ 224.230 \\ -466.398 \\ 6377.76 \\ -2089.822 \end{pmatrix} [N, Nm]$$

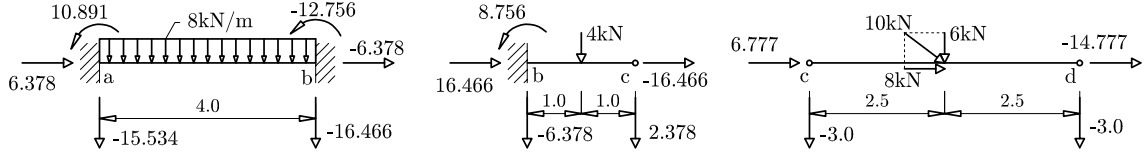
$$\{\hat{\mathbf{R}}_{bc}\} = \begin{pmatrix} 1440 & 0 & 0 & -1440 & 0 & 0 \\ 0 & 8.1 & -16.2 & 0 & -8.1 & 0 \\ 0 & -16.2 & 32.4 & 0 & 16.2 & 0 \\ -1440 & 0 & 0 & 1440 & 0 & 0 \\ 0 & -8.1 & 16.2 & 0 & 8.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 39.171 \\ 4.429 \\ -26.783 \\ 67.736 \\ -27.414 \\ 0 \end{pmatrix} = \begin{pmatrix} -41133.6 \\ 691.813 \\ -1383.626 \\ 41133.6 \\ -691.813 \\ 0 \end{pmatrix} [N, Nm]$$

$$\{\hat{\mathbf{R}}_{cd}\} = \begin{pmatrix} 207.36 & 276.48 & . & . & . & . \\ 276.48 & 368.64 & . & . & . & . \\ 0 & 0 & . & . & . & . \\ -207.36 & -276.48 & . & . & . & . \\ -276.48 & -368.64 & . & . & . & . \\ 0 & 0 & . & . & . & . \end{pmatrix} \cdot \begin{pmatrix} 67.736 \\ -27.414 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6466.314 \\ 8621.752 \\ 0 \\ -6466.314 \\ -8621.752 \\ 0 \end{pmatrix} [N, Nm]$$

$$\{\mathbf{R}_{ab}\} = \begin{pmatrix} -15533.602 \\ -6377.76 \\ 10890.89\bar{6} \\ -16466.398 \\ 6377.76 \\ -12765.488\bar{6} \end{pmatrix} \xrightarrow{\text{transform}} \{\mathbf{R}_{ab}^{\star}\} = \begin{pmatrix} 6377.76 \\ -15533.6 \\ 10890.89\bar{6} \\ -6377.76 \\ -16466.398 \\ -12765.488\bar{6} \end{pmatrix} [N, Nm]$$

$$\{\mathbf{R}_{bc}\} = \{\mathbf{R}_{bc}^{\star}\} = \begin{pmatrix} 16466.4 \\ -6378.187 \\ 8756.374 \\ -16466.4 \\ 2378.187 \\ 0 \end{pmatrix} [N, Nm]$$

$$\{\mathbf{R}_{cd}\} = \begin{pmatrix} 6466.314 \\ 3621.752 \\ 0 \\ -6466.314 \\ -13621.752 \\ 0 \end{pmatrix} \xrightarrow{\text{transform}} \{\mathbf{R}_{cd}^{\star}\} = \begin{pmatrix} 6777.19 \\ -3000 \\ 0 \\ -14777.19 \\ -3000 \\ 0 \end{pmatrix} [N, Nm]$$



Obrázek 9: Force equilibrium on the bar [kN, kNm]

8. Check force equilibrium on the bar – see Fig. 9.

Bar a – b

$$\sum F_x = 0 : 6.378 + (-6.378) = 0$$

$$\sum F_z = 0 : -15.534 + 8 \cdot 4 - 16.466 = 0$$

$$\sum M_a = 0 : 10.891 - 8 \cdot \frac{4^2}{2} + (-12.756) - (-16.466) \cdot 4 = 0$$

Bar b – c

$$\sum F_x = 0 : 16.466 + (-16.466) = 0$$

$$\sum F_z = 0 : -6.378 + 4 + 2.378 = 0$$

$$\sum M_b = 0 : 8.756 - 4 \cdot 1 - 2.378 \cdot 2 = 0$$

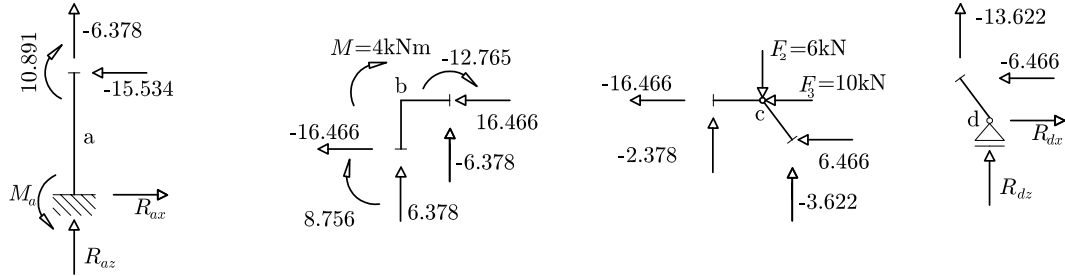
Bar c – d

$$\sum F_x = 0 : 6.777 + 8 + (-14.777) = 0$$

$$\sum F_z = 0 : -3 + 6 + (-3) = 0$$

$$\sum M_b = 0 : 6 \cdot 2.5 + (-3) \cdot 5 = 0$$

9. Check force equilibrium in nodes a, b and c and support reactions – see Fig. 10.



Obrázek 10: Force equilibrium in nodes a, b, c and d [kN, kNm]

Node a

$$\sum F_{xa} = 0 : R_{ax} - (-15.534) = 0 \Rightarrow R_{ax} = -15.534 \text{ kN}$$

$$\sum F_{za} = 0 : R_{az} + (-6.378) = 0 \Rightarrow R_{az} = 6.378 \text{ kN}$$

$$\sum M_a = 0 : M_a - 10.891 = 0 \Rightarrow M_a = 10.891 \text{ kNm}$$

Node b

$$\sum F_{xb} = 0 : -16.466 + 16.466 = 0$$

$$\sum F_{zb} = 0 : 6.378 + (-6.378) = 0$$

$$\sum M_b = 0 : 8.756 + 4 + (-12.765) = 0$$

Node c

$$\sum F_{xc} = 0 : -16.466 + 10 + 6.466 = 0$$

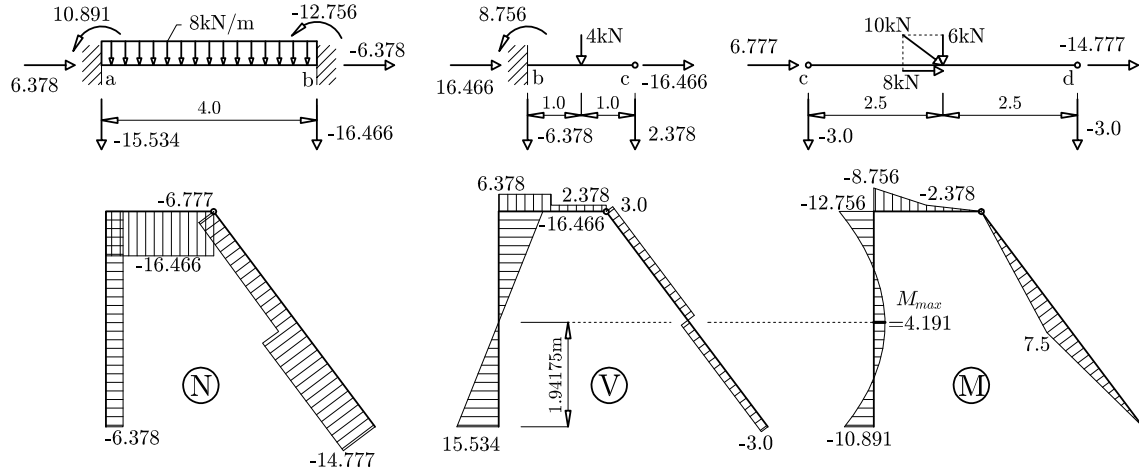
$$\sum F_{zc} = 0 : -2.378 + 6 + (-3.622) = 0$$

Node d

$$\sum F_{xd} = 0 : R_{dx} - (-6.466) = 0 \Rightarrow R_{dx} = -6.466 \text{ kN}$$

$$\sum F_{zd} = 0 : R_{dz} + (-13.622) = 0 \Rightarrow R_{dz} = -13.622 \text{ kN}$$

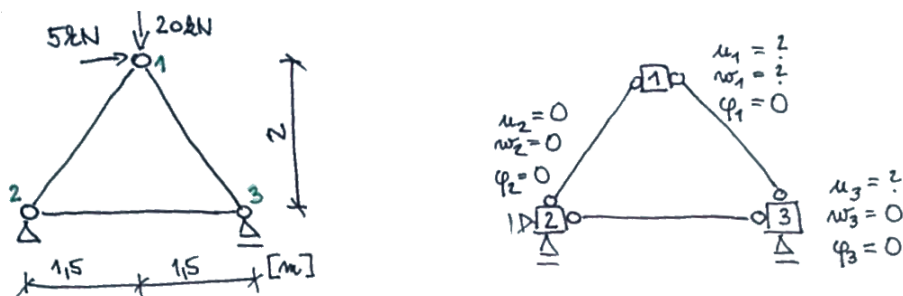
10. Internal forces diagrams (N – normal force, V – shear force, M – bending moment) – see Fig. 11.



Obrázek 11: Internal forces of the frame – N, V, M

1.3 Příhradová konstrukce

Obecnou deformační metodou vyřešte vnitřní síly na jednoduché příhradové konstrukci. Průřezová plocha prutů je $A = 15 \cdot 10^{-4} \text{ m}^2$ a modul pružnosti $E = 210 \text{ GPa}$.



Obrázek 12: Příhradová konstrukce a výpočtový model

1. Určení minimálního stupně přetvárné neurčitosti n_p a výpočtový model. Z obrázku 12 vpravo vyplývá stupeň přetvárné neurčitosti $n_p = 3$.
2. Globální vektor neznámých deformací $\mathbf{r}$ a jemu odpovídající globální vektor uzlových (styčnickových) zatížení $\mathbf{S}$.

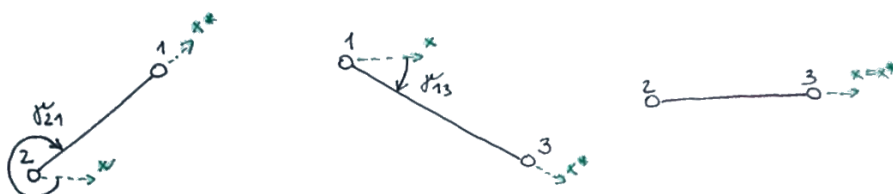
$$\{\mathbf{r}\} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \end{Bmatrix} \quad (21)$$

Vektor uzlových zatížení obsahuje síly odpovídající deformacím obsaženým ve vektoru neznámých deformací. Deformaci u_1 ve směru globální osy x ve styčniku 1 vyvolá síla 5000 N. Ve styčniku 3 se nenachází žádná síla ve směru osy x , která by způsobila posun u_3 , proto 0.

$$\{\mathbf{S}\} = \begin{Bmatrix} 5000 \\ 20000 \\ 0 \end{Bmatrix} [\text{N}] \quad (22)$$

3. Výpočet globálních matic tuhosti pro jednotlivé pruty (volím pruty 21, 13, 23) a sestavení globální matice tuhosti soustavy.

Protože jsou všechny pruty oboustranně kloubově připojené (kloub-kloub), budou následující matice tuhosti zjednodušeny na matice 4×4 . Pro výpočet matic tuhosti šikmých prutů použijeme tabulku 8.2d.



Obrázek 13: Úhly natočení jednotlivých prutů

Parametry pro prut 21 – $l = 2.5 \text{ m}$, $s = -\frac{4}{5}$ a $c = \frac{3}{5}$.

$$[\mathbf{k}_{21}] = \begin{pmatrix} u_2 & w_2 & u_1 & w_1 \\ \cdot & \cdot & -45.36 & 60.48 \\ \cdot & \cdot & 60.48 & -80.64 \\ \cdot & \cdot & 45.36 & -60.48 \\ \cdot & \cdot & -60.48 & 80.64 \end{pmatrix} \begin{pmatrix} u_2 \\ w_2 \\ u_1 \\ w_1 \end{pmatrix} \cdot 10^6 \left[\frac{\text{N}}{\text{m}} \right]$$

Parametry pro prut 13 – $l = 2.5 \text{ m}$, $s = \frac{4}{5}$ a $c = \frac{3}{5}$.

$$[k_{13}] = \begin{pmatrix} & \begin{matrix} u_1 \\ 45.36 \\ 60.48 \\ -45.36 \\ -60.48 \end{matrix} & \begin{matrix} w_1 \\ 60.48 \\ 80.64 \\ -60.48 \\ -80.64 \end{matrix} & \begin{matrix} u_3 \\ -45.36 \\ -60.48 \\ 45.36 \\ 60.48 \end{matrix} & \begin{matrix} w_3 \\ . \\ . \\ . \\ . \end{matrix} \end{pmatrix} \begin{pmatrix} u_1 \\ w_1 \\ u_3 \\ w_3 \end{pmatrix} \cdot 10^6 \left[\frac{\text{N}}{\text{m}} \right]$$

Lokální i globální souřadný systém prutu 23 jsou totožné, proto můžeme použít tabulku 8.3d pro lokální matice tuhosti prutu.

$$[k_{23}] = \begin{pmatrix} & u_2 & w_2 & \begin{matrix} u_3 \\ -105 \\ 0 \\ 105 \\ 0 \end{matrix} & w_3 \end{pmatrix} \begin{pmatrix} u_2 \\ w_2 \\ u_3 \\ w_3 \end{pmatrix} \cdot 10^6 \left[\frac{\text{N}}{\text{m}} \right]$$

Globální matici tuhosti soustavy prutů získáme sečtením příslušných tuhostí pro naše neznámé deformace (šedá políčka v maticích tuhostí). Vznikne tak matice o velikosti $n_p \times n_p$.

$$[k] = \left(\begin{array}{cc|c} 45.36 & -60.48 & -45.36 \\ +45.36 & +60.48 & \\ \hline -60.48 & 80.64 & -60.48 \\ +60.48 & +80.64 & \\ \hline -45.36 & -60.48 & 45.36 \\ & & +105 \end{array} \right) \cdot 10^6 = \begin{pmatrix} 90.72 & 0 & -45.36 \\ 0 & 161.28 & -60.48 \\ -45.36 & -60.48 & 150.36 \end{pmatrix} \cdot 10^6 \quad (23)$$

4. Řešení soustavy lineárních rovnic (např. Gaussova eliminační metoda):

$$[k] \cdot \{r\} = \{F\} \quad (24)$$

Protože se jedná o příhradovou konstrukci, kde je zatížení soustředěno ve styčnicích, vektor sil $\{F\}$ je roven vektoru uzlových zatížení.

$$\{F\} = \{S\} \quad (25)$$

$$\begin{pmatrix} 90.72 & 0 & -45.36 \\ 0 & 161.28 & -60.48 \\ -45.36 & -60.48 & 150.36 \end{pmatrix} \begin{pmatrix} u_1 \\ w_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 5000 \\ 20000 \\ 0 \end{pmatrix} [\text{N}] \quad (26)$$

Vyřešením soustavy získáváme neznámé deformace konstrukce:

$$\begin{pmatrix} u_1 \\ w_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1.027337 \\ 1.597222 \\ 0.952381 \end{pmatrix} \cdot 10^{-4} [\text{m}] \quad (27)$$

5. Výpočet globálních vektorů výsledných koncových sil a jejich transformace do lokálního souřadného systému jednotlivých prutů. Výsledné koncové síly jsou rovny sekundárním koncovým silám, protože konstrukce neobsahuje zatížení na prutech (žádné vektory primárních koncových sil).

$$R_{ab} = \hat{R}_{ab} \quad (28)$$

$$[R_{21}] = \begin{pmatrix} . & . & -45.36 & 60.48 \\ . & . & 60.48 & -80.64 \\ . & . & 45.36 & -60.48 \\ . & . & -60.48 & 80.64 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1.027337 \\ 1.597222 \end{pmatrix} \cdot 10^2 = \begin{pmatrix} 5000 \\ -6666.\bar{6} \\ -5000 \\ 6666.\bar{6} \end{pmatrix} [\text{N}]$$

Pozor na vliv zaokrouhlování! Při použití $\{r_{21}\} = \{0, 0, 103, 160\}^T \cdot 10^{-6}$ dostaneme vektor výsledných koncových sil $\{R_{21}\} = \{5004.72, -6672.96, -5004.72, 6672.96\}^T$.

$$[R_{13}] = \begin{pmatrix} 45.36 & 60.48 & -45.36 & . \\ 60.48 & 80.64 & -60.48 & . \\ -45.36 & -60.48 & 45.36 & . \\ -60.48 & -80.64 & 60.48 & . \end{pmatrix} \cdot \begin{Bmatrix} 1.027337 \\ 1.597222 \\ 0.952381 \\ 0 \end{Bmatrix} \cdot 10^2 = \begin{Bmatrix} 10000 \\ 13333.\bar{3} \\ -10000 \\ -13333.\bar{3} \end{Bmatrix} [N]$$

Pozor na vliv zaokrouhlování! Při použití $\{r_{13}\} = \{103, 160, 95, 0\}^T \cdot 10^{-6}$ dostaneme vektor výsledných koncových sil $\{R_{13}\} = \{10039.68, 13386.24, -10039.68, -13386.34\}^T$.

$$[R_{23}] = \begin{pmatrix} . & . & -105 & . \\ . & . & 0 & . \\ . & . & 105 & . \\ . & . & 0 & . \end{pmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0.952381 \\ 0 \end{Bmatrix} \cdot 10^2 = \begin{Bmatrix} -10000 \\ 0 \\ 10000 \\ 0 \end{Bmatrix} [N]$$

Pozor na vliv zaokrouhlování! Při použití $\{r_{23}\} = \{0, 0, 95, 0\}^T \cdot 10^{-6}$ dostaneme vektor výsledných koncových sil $\{R_{23}\} = \{-9975, 0, 9975, 0\}^T$.

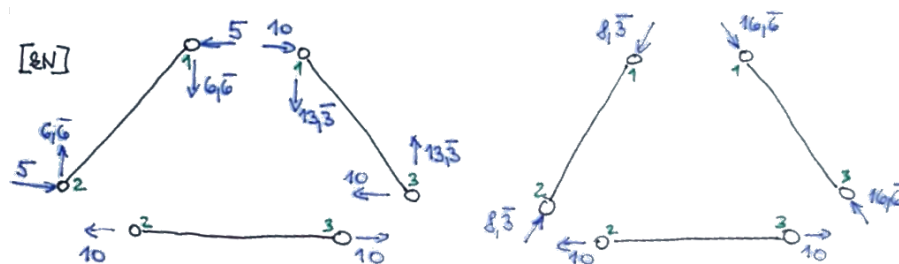
Síly v globálních vektorech výsledných koncových sil můžeme vykreslit přímo do příslušných styčníků a pomocí rozkladu sil získat osové síly na prutech, což je to samé jako provedení transformace:

$$R_{ab}^* = T_{ab} R_{ab} \quad (29)$$

$$\{R_{21}^*\} = \begin{pmatrix} 3/5 & -4/5 & 0 & 0 \\ 4/5 & 3/5 & 0 & 0 \\ 0 & 0 & 3/5 & -4/5 \\ 0 & 0 & 4/5 & 3/5 \end{pmatrix} \begin{Bmatrix} 5000 \\ -6666.\bar{6} \\ -5000 \\ 6666.\bar{6} \end{Bmatrix} = \begin{Bmatrix} 8333.\bar{3} \\ 0 \\ -8333.\bar{3} \\ 0 \end{Bmatrix}$$

$$\{R_{21}^*\} = \begin{pmatrix} 3/5 & 4/5 & 0 & 0 \\ -4/5 & 3/5 & 0 & 0 \\ 0 & 0 & 3/5 & 4/5 \\ 0 & 0 & -4/5 & 3/5 \end{pmatrix} \begin{Bmatrix} 10000 \\ 13333.\bar{3} \\ -10000 \\ -13333.\bar{3} \end{Bmatrix} = \begin{Bmatrix} 16666.\bar{6} \\ 0 \\ -16666.\bar{6} \\ 0 \end{Bmatrix}$$

6. Vykreslení vnitřních sil na konstrukci.

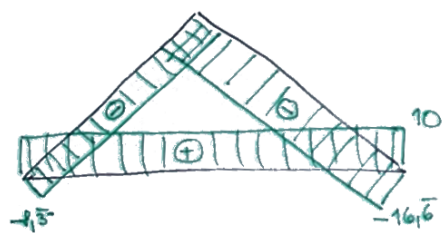


Obrázek 14: Globální a lokální výsledné koncové síly na jednotlivých prutech

Pozn. 1: Protože se jedná o jednoduchou staticky určitou konstrukci, můžete provést kontrolu výsledků pomocí styčnickové metody (viz Základy stavební mechaniky).

Pozn. 2: Pokud by se měly počítat pouze deformace, stačí vyčíslvat pouze šedá políčka matic tuhosti a primárních vektorů. Pokud se počítá celý příklad, je potřeba vyčíslit celé sloupce matic tuhosti náležící neznámým deformacím a celé primární vektory.

Pozn. 3: Pro výpočty sin a cos je obvykle (školní příklady) výhodnější zápis poměrem stran pravoúhlého trojúhelníku než přepočít přes úhel (obvykle pěkné zlomky $\Rightarrow$ menší zaokrouhlovací chyby).



Obrázek 15: Normálové síly příhradové konstrukce