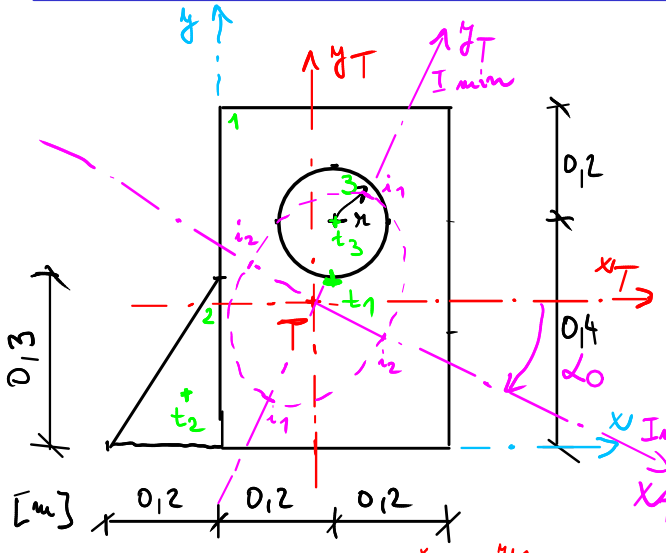
Průřezové charakteristiky složených obrazců



$r = 0,1\ m$ - otvor

- plocha

$A_1 = 0,4 \cdot 0,6 = 0,24\ m^2$

$A_2 = \frac{1}{2}\, 0,2 \cdot 0,3 = 0,03\ m^2$

$A_3 = \pi r^2 = \pi \cdot 0,1^2 = 0,03142\ m^2$ (otvor)

$A = A_1 + A_2 \overset{\text{otvor}}{\ominus} A_3 = 0,24 + 0,03 - 0,03142 = \underline{0,23858\ m^2}$

- těžiště $\quad t_1[\overset{x_{t1}}{0,2};\overset{y_{t1}}{0,3}] \quad t_2[\overset{x_{t2}}{-0,06};\overset{y_{t2}}{0,1}] \quad t_3[\overset{x_{t3}}{0,2};\overset{y_{t3}}{0,4}]$

- statický moment

$$S_x = \sum_{i=1}^{n} S_{xi} = \sum_{i=1}^{n} A_i y_{ti} = A_1 \cdot y_{t1} + A_2 y_{t2} \overset{\text{otvor}}{\ominus} A_3 y_{t3} =$$

$$= 0,24 \cdot 0,3 + 0,03 \cdot 0,1 - 0,03142 \cdot 0,4 = 62,432 \cdot 10^{-3}\ m^3$$

$$S_y = \sum_{i=1}^{n} A_i x_{ti} = A_1 x_{t1} + A_2 x_{t2} \overset{\text{otvor}}{\ominus} A_3 x_{t3} = 0,24 \cdot 0,2 + 0,03 \cdot (-0,06) - 0,03142 \cdot 0,2 =$$

$$= 39,716 \cdot 10^{-3}\ m^3$$

$$x_T = \frac{S_y}{A} = \frac{39,716 \cdot 10^{-3}}{0,23858} = \underline{0,166\ m}$$

$$y_T = \frac{S_x}{A} = \frac{62,432 \cdot 10^{-3}}{0,23858} = \underline{0,262\ m}$$

$T[0,166;\ 0,262]$

- moment setrvačnosti a deviační moment (centrální)

Steinerova věta - moment setrvačnosti k libovolné ose

$$I_x = I_{xt} + A c^2 \qquad I_y = I_{yt} + A d^2 \qquad D_{xy} = D_{x_t y_t} + A c d$$

$$c = y_t - y \qquad d = x_t - x$$

$$I_{x_T} = \sum_{i=1}^{n} (I_{xti} + A_i c_i^2) = \frac{1}{12} b_1 h_1^3 + A_1 \cdot (y_{t1} - y_T)^2 + \frac{1}{36} b_2 h_2^3 + A_2 (y_{t2} - y_T)^2 \overset{\text{otvor}}{\ominus} \left[\frac{\pi r^4}{4} + A_3 (y_{t3} - y_T)^2\right]$$

$$= \frac{1}{12} 0,4 \cdot 0,6^3 + 0,24 \cdot (0,3 - 0,262)^2 + \frac{1}{36} 0,2 \cdot 0,3^3 + 0,03 (0,1 - 0,262)^2 - \frac{\pi \cdot 0,1^4}{4} -$$

$$- 0,03142 \cdot (0,4 - 0,262)^2 = \underline{7,804 \cdot 10^{-3}\ m^4}$$

$$I_{y_T} = \sum_{i=1}^{n} (I_{yti} + A_i d_i^2) = \frac{1}{12} h_1 b_1^3 + A_1 (x_{t1} - x_T)^2 + \frac{1}{36} h_2 b_2^3 + A_2 (x_{t2} - x_T)^2 \overset{\text{otvor}}{\ominus} \left[\frac{\pi r^4}{4} + A_3 (x_{t3} - x_T)^2\right] =$$

$$= \frac{1}{12} 0,6 \cdot 0,4^3 + 0,24 (0,2 - 0,166)^2 + \frac{1}{36} 0,3 \cdot 0,2^3 + 0,03 (-0,06 - 0,166)^2 - \frac{\pi 0,1^4}{4} -$$

$$- 0,03142 (0,2 - 0,166)^2 = \underline{5,053 \cdot 10^{-3}\ m^4}$$

$D_{x_t y_t}$ ⊕ ⊖

$$D_{x_T y_T} = \sum_{i=1}^{n} (D_{x_{ti} y_{ti}} + A_i c_i d_i) = D_{x_{t1} y_{t1}} + A_1 (y_{t1} - y_T)(x_{t1} - x_T) + \underbrace{D_{x_{t2} y_{t2}}}_{\oplus \frac{b_2^2 h_2^2}{72}} + A_2 (y_{t2} - y_T)(x_{t2} - x_T)$$

$$\overset{\text{otvor}}{\ominus} \left[D_{x_{t3} y_{t3}} + A_3 (y_{t3} - y_T)(x_{t3} - x_T)\right]$$

$$= 0 + 0{,}24\,(0{,}3-0{,}262)(0{,}2-0{,}166) + \frac{0{,}2^2\cdot 0{,}3^2}{42} + 0{,}03\,(0{,}1-0{,}262)(-0{,}06-0{,}166) -$$

$$- 0 - 0{,}03142\,(0{,}4-0{,}262)(0{,}2-0{,}166) = \underline{1{,}343\cdot 10^{-3}\ m^4}$$

- hlavní momenty setrvačnosti ($\Rightarrow D_{12}=0$)

$$I_{max\atop min} = I_{1,2} = \frac{I_x+I_y}{2} \pm \frac{1}{2}\sqrt{(I_x-I_y)^2+4D_{xy}^2} =$$

$$= \frac{7{,}807\cdot 10^{-3}+5{,}053\cdot 10^{-3}}{2} \pm \frac{1}{2}\sqrt{\left[(7{,}807-5{,}053)\cdot 10^{-3}\right]^2+4\cdot(1{,}343\cdot 10^{-3})^2} =$$

$I_1 > I_2$

$$\underline{I_1 = 8{,}353\cdot 10^{-3}\ m^4}$$

$$\underline{I_2 = 4{,}507\cdot 10^{-3}\ m^4}$$

KONTROLA:

1. invariant $\quad I_{xT}+I_{yT} = I_1+I_2 = I_x+I_y$

$$7{,}807+5{,}053 = 8{,}353+4{,}507$$

$$12{,}86 = 12{,}86 \ \checkmark$$

2. invariant $\quad I_{xT}I_{yT} - D_{xTyT}^2 = I_xI_y - D_{xy}^2 = I_1\cdot I_2\ (-0)$

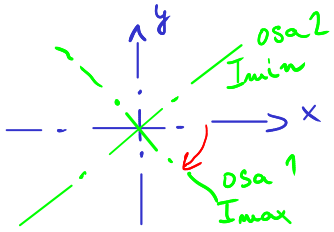
$$7{,}807\cdot 5{,}053 - 1{,}343^2 = 8{,}353\cdot 4{,}507$$

$$37{,}645 = 37{,}645 \ \checkmark$$

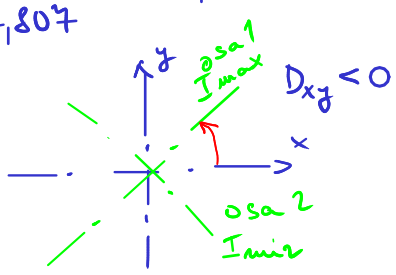
- natočení hlavních os

$$tg\,2\alpha_0 = \frac{2D_{xy}}{I_y-I_x} = \frac{2\cdot 1{,}343}{5{,}053-7{,}807} = -0{,}9753 \Rightarrow \alpha_0 = \underline{-22{,}142^\circ}$$

$D_{xy}>0$



$D_{xy}<0$



- poloměr setrvačnosti

$$i_x = \sqrt{\frac{I_x}{A}} \qquad i_y = \sqrt{\frac{I_y}{A}}$$

$$i_x = \sqrt{\frac{7{,}807\cdot 10^{-3}}{0{,}23858}} = 0{,}1809\ m \qquad i_y = \sqrt{\frac{5{,}053\cdot 10^{-3}}{0{,}23858}} = 0{,}1455\ m$$

- hlavní centrální poloměry setrvačnosti

$$i_1 = \sqrt{\frac{I_1}{A}} = \sqrt{\frac{8{,}353\cdot 10^{-3}}{0{,}23858}} = \underline{0{,}187\ m} \qquad i_2 = \sqrt{\frac{I_2}{A}} = \sqrt{\frac{4{,}507\cdot 10^{-3}}{0{,}23858}} = \underline{0{,}137\ m}$$

- vlastní čísla

$$\begin{bmatrix} I_x & -D_{xy} \\ -D_{xy} & I_y \end{bmatrix} = \begin{bmatrix} 7{,}807 & -1{,}343 \\ -1{,}343 & 5{,}053 \end{bmatrix}$$

$$\begin{vmatrix} I_{xT}-\lambda & D_{xTyT} \\ D_{xTyT} & I_{yT}-\lambda \end{vmatrix} = (I_{xT}-\lambda)(I_{yT}-\lambda) \ominus D_{xTyT}^2$$

$$10^{-3}\left[(7{,}807-\lambda)(5{,}053-\lambda) - 1{,}343^2\right] = 0$$

$$10^{-3}\left[\lambda^2 - 12{,}86\,\lambda + 37{,}645\right] = 0$$

$$D = b^2-4ac = 14{,}7496\cdot 10^{-3}$$

$$\lambda_{1,2} = I_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \begin{cases} I_1 = 8{,}353 \cdot 10^{-3}\ m^4 \\ I_2 = 4{,}507 \cdot 10^{-3}\ m^4 \end{cases}$$

• vlastní vektory $(A - \lambda I)u = \emptyset$

$I_1 = 8{,}353 \cdot 10^{-3}\ m^4$

$$\begin{bmatrix} 7{,}807 - 8{,}353 & -1{,}343 \\ -1{,}343 & 5{,}053 - 8{,}353 \end{bmatrix} \begin{Bmatrix} {}^1u_1 \\ {}^1u_2 \end{Bmatrix} = \emptyset$$

$$\begin{bmatrix} -0{,}546 & -1{,}343 \\ -1{,}343 & -3{,}3 \end{bmatrix} \ominus \begin{matrix} 1{,}343 \\ 0{,}546 \end{matrix} \rightarrow \begin{bmatrix} -0{,}546 & -1{,}343 \\ 0 & \sim 0 \end{bmatrix} \Rightarrow {}^1u_1 = -2{,}4597t \quad \Rightarrow {}^1u_2 = t$$

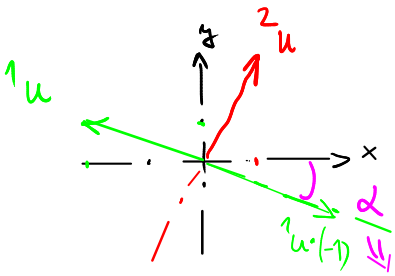
$${}^1u = \begin{Bmatrix} -2{,}4597 \\ 1 \end{Bmatrix} t \qquad \|{}^1u\| = 2{,}6552 \qquad {}^1u = \frac{{}^1u}{\|{}^1u\|} = \begin{Bmatrix} -0{,}926 \\ 0{,}377 \end{Bmatrix} t$$

$I_2 = 4{,}507 \cdot 10^{-3}\ m^4$

$$\begin{bmatrix} 7{,}807 - 4{,}507 & -1{,}343 \\ -1{,}343 & 5{,}053 - 4{,}507 \end{bmatrix} \begin{Bmatrix} {}^2u_1 \\ {}^2u_2 \end{Bmatrix} = \emptyset$$

$$\begin{bmatrix} 3{,}3 & -1{,}343 \\ -1{,}343 & 0{,}546 \end{bmatrix} \rightarrow \begin{bmatrix} 3{,}3 & -1{,}343 \\ 0 & \sim 0 \end{bmatrix} \Rightarrow {}^1u_2 = 0{,}40697t \quad \Rightarrow {}^2u_2 = t$$

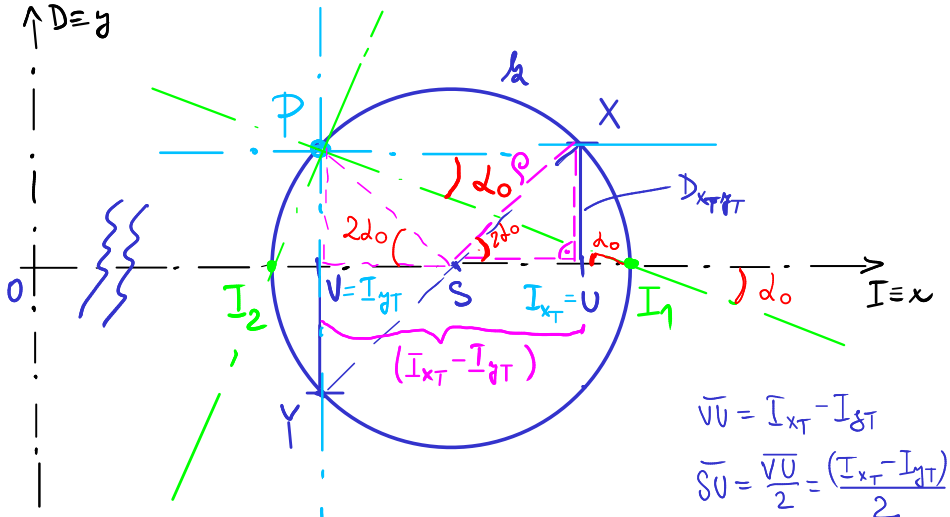
$${}^2u = \begin{Bmatrix} 0{,}40697 \\ 1 \end{Bmatrix} t \qquad \|{}^2u\| = 1{,}0796 \qquad {}^2u = \frac{{}^2u}{\|{}^2u\|} = \begin{Bmatrix} 0{,}377 \\ 0{,}926 \end{Bmatrix} t$$



$$\operatorname{tg}\alpha = \frac{0{,}377}{0{,}926} = 22{,}15° \checkmark$$

(se znaménky ${}^1u = \begin{Bmatrix} 2{,}4597 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0{,}926 \\ -0{,}377 \end{Bmatrix}$, $\operatorname{tg}\alpha = \frac{-0{,}377}{0{,}926} = -22{,}15°$)

• Mohrova kružnice



1) osy $I \perp D$

2) $I_{x_T} = \overline{OU}$, $I_{y_T} = \overline{OV}$

3) $\overline{UX} = \overline{VY} = D_{x_T y_T}$

$\overline{UX}$ nad osou I pro $\oplus D_{xy}$

$\overline{VY}$ opačně

4) $\overline{XY}$ ≡ průměr Mohrovy kružnice $k(S, \rho)$

$I \cap \overline{XY} \equiv S$

poloměr ρ

$$\overline{OS} = \frac{I_{x_T} + I_{y_T}}{2}$$

5) bodem X vedu $\parallel s_x\,(I) \cap k \equiv P$

Y $\parallel y\,(D) \cap k \equiv P$

6) $k \cap I = I_1$ a I_2

7) $\overline{PI_1}$ a $\overline{PI_2}$ = hlavní osy

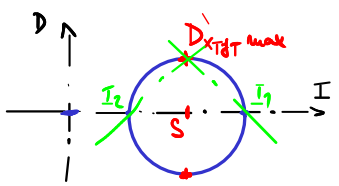
$\overline{VU} = I_{x_T} - I_{y_T}$

$\overline{SU} = \frac{\overline{VU}}{2} = \frac{(I_{x_T} - I_{y_T})}{2}$

$$\rho = \sqrt{\overline{SU}^2 + \overline{UX}^2} = \sqrt{\left(\frac{I_{x_T} - I_{y_T}}{2}\right)^2 + D_{x_T y_T}^2} =$$

$$= \boxed{\frac{1}{2}\sqrt{(I_{x_T} - I_{y_T})^2 + 4 D_{x_T y_T}^2}}$$

$$I_{1,2} = \overline{OS} \pm \rho \qquad D_{xy}(\text{při } I_1, I_2) = 0$$

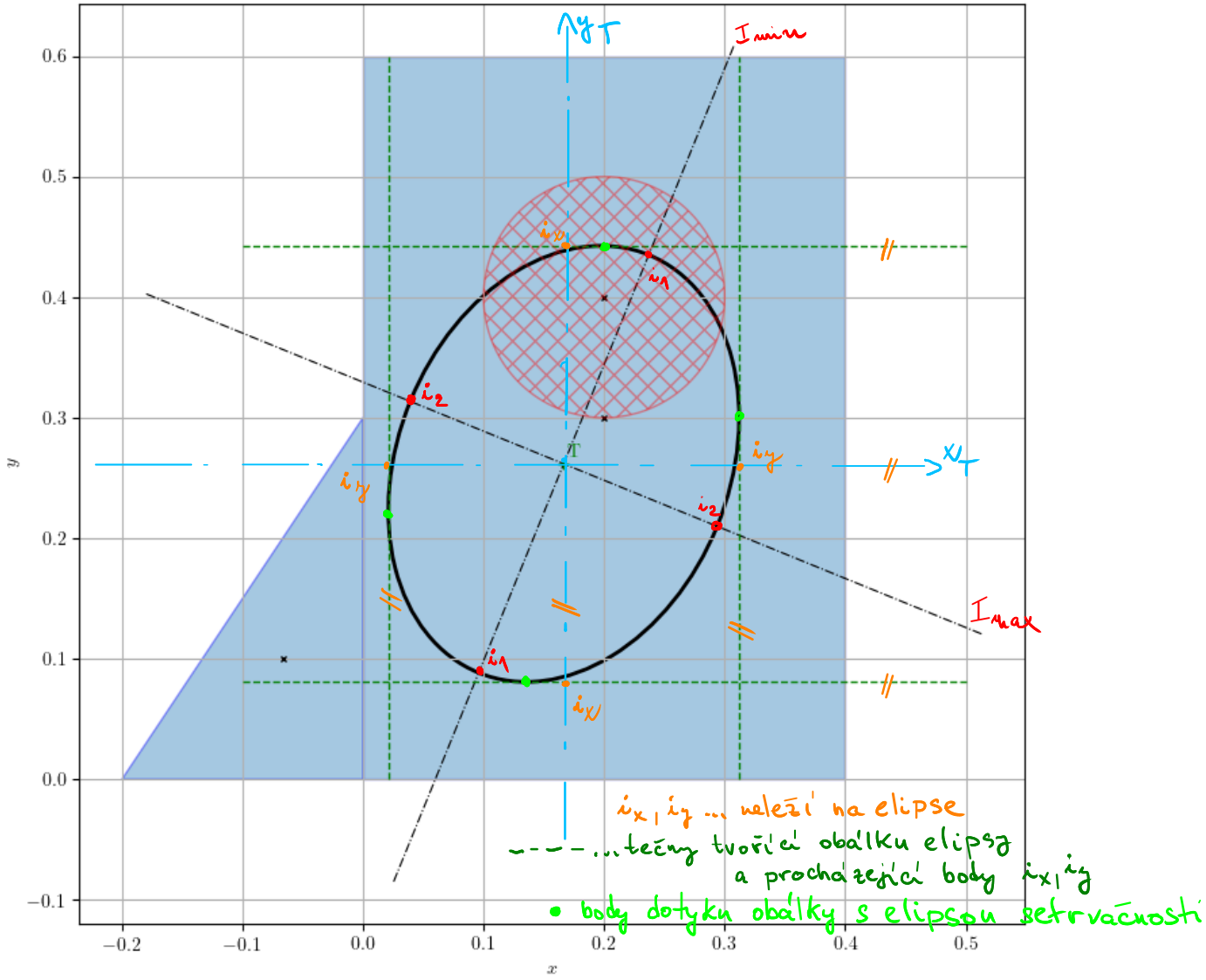


$D_{x_T y_T} = max$

$I'_{x_T} = I'_{y_T} = \frac{I_{x_T} + I_{y_T}}{2}$

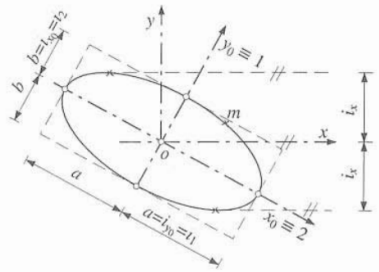
$\alpha_0 = 45°$ od hlavních os

$$\operatorname{tg} 2\alpha_0 = \frac{\overline{UX}}{\overline{SU}} = \frac{2 D_{x_T y_T}}{I_{x_T} - I_{y_T}} \quad \text{(bez znaménka)}$$



5.6. Elipsa setrvačnosti

Proložíme-li daným bodem o v rovině obrazce osy setrvačnosti různými směry, přísluší každé ose, např. x, jistý moment setrvačnosti I_x (5.2) a tedy i jistá velikost poloměru setrvačnosti i_x (5.39). Vedeme-li přímky rovnoběžné s osami setrvačnosti ve vzdálenostech příslušných poloměrům setrvačnosti, obalí tyto přímky křivku zvanou *elipsa setrvačnosti pro daný bod* o (obr. 5.14).



Obr. 5.14. Elipsa setrvačnosti obrazce pro bod o

Elipsa setrvačnosti má v souřadnicové soustavě x, y rovnici

$$I_x x^2 + I_y y^2 - 2D_{xy} xy = \frac{I_x I_y - D_{xy}^2}{A}, \quad (5.45)$$

v níž x, y jsou souřadnice jejího libovolného bodu m, A je obsah obrazce a I_x, I_y, D_{xy} jsou kvadratické momenty. Odvození rovnice (5.45) je provedeno např. v práci [20].